Back to dynamics

\[ \nabla \cdot B = 0 \Rightarrow B = \nabla \times A \]

remain true

But now instead of \( \nabla \times E = 0 \) we have

\[ \nabla \times E + \frac{1}{c} \frac{\partial B}{\partial t} = 0 \]

\[ \Rightarrow \nabla \times E + \frac{1}{c} \frac{\partial B}{\partial t} = 0 \]

\[ \Rightarrow \nabla \times (E + \frac{1}{c} \frac{\partial A}{\partial t}) = 0 \]

\[ \Rightarrow \text{there exists a scalar potential } \phi \text{ such that} \]

\[ E + \frac{1}{c} \frac{\partial A}{\partial t} = -\nabla \phi \quad \text{or} \quad E = -\nabla \phi - \frac{1}{c} \frac{\partial A}{\partial t} \]

Gauss's law for electric field now becomes

\[ \nabla \cdot E = \frac{4\pi}{c} \rho = -\nabla^2 \phi - \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot A) = \frac{4\pi}{c} \rho \]

\[ \nabla^2 \phi + \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot A) = -\frac{4\pi}{c} \rho \]

Gauss law in terms of electromagnetic potentials

Ampère's law becomes

\[ \nabla \times B = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial E}{\partial t} \]

\[ \nabla \times (\nabla \times A) = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial}{\partial t} \nabla \phi - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} \]
\[
-\nabla^2 \vec{A} + \vec{V} (\vec{V} \cdot \vec{A}) = \frac{4 \pi}{\varepsilon} \frac{\vec{J}}{\varepsilon} - \frac{1}{\varepsilon} \frac{2}{\varepsilon} \left( \frac{\vec{V} \phi}{\varepsilon} + \frac{1}{\varepsilon} \frac{\partial \vec{A}}{\partial t} \right)
\]

\[
-\nabla^2 \vec{A} + \frac{1}{\varepsilon} \frac{\partial \vec{A}}{\partial t} = \frac{4 \pi}{\varepsilon} \frac{\vec{J}}{\varepsilon} - \vec{V} (\vec{V} \cdot \vec{A} + \frac{1}{\varepsilon} \frac{\partial \phi}{\partial t})
\]

**Gauge invariance**

As before, we can always construct \( \vec{A}' = \vec{A} + \vec{V} \chi \), for any scalar function \( \chi \), that gives the same \( \vec{E} \).

But since \( \vec{A} \) now also enters expression for \( \vec{E} \), we need to make sure that if we change \( \vec{A} \) to \( \vec{A}' \), we must make some corresponding change \( \phi \) to \( \phi' \) so that \( \vec{E} \) does not change.

\[
\begin{bmatrix}
\vec{A}' = \vec{A} + \vec{V} \chi \\
\phi' = \phi - \frac{1}{\varepsilon} \frac{\partial \chi}{\partial t}
\end{bmatrix}
\]

**Gauge transformation**

For any scalar \( \chi \), the above \( \vec{A}' \) ad \( \phi' \) give the same values of \( \vec{E} \) ad \( \vec{B} \) as \( \vec{A} \) ad \( \phi \).

**Proof:**

\[
\vec{V} \times \vec{A}' = \vec{V} \times \vec{A} + \vec{V} \times \vec{V} \chi = \vec{V} \times \vec{A} = \vec{B}
\]

\[
(-\vec{V} \phi' - \frac{1}{\varepsilon} \frac{\partial \vec{A}'}{\partial t}) = -\vec{V} \phi + \frac{1}{\varepsilon} \frac{\vec{V} \chi}{\partial t} - \frac{1}{\varepsilon} \frac{\partial \vec{A}}{\partial t} - \frac{1}{\varepsilon} \frac{2}{\varepsilon} \frac{\vec{V} \chi}{\partial t}
\]

\[
= (-\vec{V} \phi - \frac{1}{\varepsilon} \frac{\partial \vec{A}}{\partial t}) = \vec{E}
\]

As before, we can fix the gauge by imposing some additional constraint on \( \vec{A} \) ad \( \phi \). There are two popular choices:
1) Lorentz Gauge

Gauge constraint: require \( \frac{1}{c} \frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0 \)

Then Gauss' law becomes

\[ \nabla^2 \phi + \frac{1}{c^2} \frac{\partial}{\partial t} \left( \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = -4\pi \rho \]

\[ \Rightarrow \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -4\pi \rho \]

Ampere's law becomes

\[ -\nabla^2 \vec{A} + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c} \vec{J} - \vec{\nabla} \left( \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) \]

\[ \Rightarrow \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c} \vec{J} \]

The combination \( -\nabla^2 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \equiv \Delta^2 \) is the wave equation operator.

In Lorentz gauge, \( \vec{A} \) and \( \phi \) satisfy the inhomogeneous wave equations:

\[ \Box^2 \vec{A} = \frac{4\pi}{c} \vec{J} \]

\[ \Box^2 \phi = 4\pi \rho \]

when \( \vec{J} = 0, \rho = 0 \) electromagnetic waves are solution!
prove that we can always find \( \hat{A} \) and \( \phi \) that satisfy the Lorentz gauge condition.

Suppose \( \nabla \times \hat{A} = \vec{B} \) and \( -\nabla \phi - \frac{1}{c} \frac{\partial \hat{A}}{\partial t} = \vec{E} \)

but \( \frac{1}{c} \frac{\partial \phi}{\partial t} + \nabla \cdot \hat{A} = D(\vec{r}, t) \neq 0 \)

Construct \( \hat{A}' = \hat{A} + \nabla \chi \)

\( \phi' = \phi - \frac{1}{c} \frac{\partial \chi}{\partial t} \)

by gauge invariance we know \( \hat{A}' \) and \( \phi' \) give the same \( \vec{E} \) and \( \vec{B} \) as before.

Now: \( \nabla \cdot \hat{A}' + \frac{1}{c} \frac{\partial \phi'}{\partial t} = \nabla \cdot \hat{A} + \nabla^2 \chi + \frac{1}{c} \frac{\partial \phi}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \chi}{\partial t^2} \)

\( = D - \Delta^2 \chi \)

So \( \hat{A}' \) and \( \phi' \) will be in the Lorentz gauge provided we choose \( \chi(\vec{r}, t) \) such that

\( \Delta^2 \chi = D \leftarrow \text{inhomogeneous wave equation} \)

Just like there is always a solution to Poisson's eq. \( \Delta^2 \phi = f \), so there is always a solution to the inhomogeneous wave equation, hence we can always find a \( X(\vec{r}, t) \) that transforms to the Lorentz gauge.
Note: Lorentz gauge condition does not uniquely determine $\mathbf{A}$ and $\phi$. If one constructs $\mathbf{A}'$ and $\phi'$ obeying Lorentz gauge condition, and then constructs

$$\mathbf{A}' = \mathbf{A} + \nabla \chi$$
$$\phi' = \phi - \frac{1}{c} \frac{\partial}{\partial t} \chi$$

then $\mathbf{A}'$ and $\phi'$ will also be in Lorentz gauge provided $\nabla^2 \chi = 0$ (proof left to reader).

2) Coulomb gauge

Gauge constraint: require $\mathbf{\nabla} \cdot \mathbf{A} = 0$

If $\mathbf{A}$ is in the Coulomb gauge, then $\mathbf{A}' = \mathbf{A} + \nabla \chi$ will also be in Coulomb gauge provided $\nabla^2 \chi = 0$.

Then Gauss' law becomes

$$\nabla^2 \phi + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} (\mathbf{\nabla} \cdot \mathbf{A}) = -4\pi \rho$$

$$\Rightarrow \boxed{\nabla^2 \phi = -4\pi \rho} \text{ same as electrostatics!}$$

$$\Rightarrow \phi(\mathbf{r}) = \int d^3 r' \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|}$$

no matter what motion the source $\rho(\mathbf{r}, t)$ has! $\phi$ is given by the instantaneous Coulomb potential even though electromagnetic fields have a finite velocity of propagation $c$!
Ampere's Law becomes:

\[- \nabla^2 \vec{A} + \frac{1}{c^2} \frac{\partial \vec{A}}{\partial t} = \frac{1}{\varepsilon_0} \frac{\partial \vec{j}}{\partial t} - \nabla \left( \nabla \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \right)\]

\[\nabla^2 \vec{A} = \frac{4\pi}{\varepsilon_0} \vec{j} - \frac{1}{c} \nabla \left( \frac{\partial \phi}{\partial t} \right) \text{ since } \nabla \cdot \vec{A} = 0\]

Now we use the solution for \(\phi\) in the Coulomb gauge to write:

\[\nabla \left( \frac{\partial \phi}{\partial t} \right) = \nabla \left[ \int d^3r' \frac{\partial \phi(r',t)}{\partial t} \frac{1}{(\vec{r} - \vec{r}')} \right]\]

\[= -\nabla \left[ \int d^3r' \frac{\nabla' \cdot \vec{f}(r',t)}{1 - \vec{r} \cdot \vec{r}'} \right]\]

The last step follows from conservation of charge \(\nabla \cdot \vec{j} = -\frac{\partial \phi}{\partial t}\).

To see the meaning of this term, recall (and we will soon demonstrate explicitly) that any vector function \(\vec{f}(\vec{r},t)\) can always be written as the sum of a curlfree part and a divergenceless part:

\[\vec{f} = \vec{f}_\parallel + \vec{f}_\perp\]

where \(\nabla \times \vec{f}_\parallel = 0\) curlfree \(\nabla \cdot \vec{f}_\perp = 0\) divergenceless.

When \(\nabla \cdot \vec{f}\) and \(\nabla \times \vec{f}\) are localized functions that vanish as \(\vec{r} \to \infty\), we have for solutions (proof to follow):

\[\vec{f}_\parallel (\vec{r}) = -\frac{1}{4\pi} \nabla \int d^3r' \frac{\nabla' \cdot \vec{f}(\vec{r}')}{(\vec{r} - \vec{r}')^3}\]

\[\vec{f}_\perp (\vec{r}) = \frac{1}{4\pi} \nabla \times \int d^3r' \frac{\nabla' \times \vec{f}(\vec{r}')}{(\vec{r} - \vec{r}')^3}\]
The electric part is also called the \underline{longitudinal} part
the divergenceless part is also called the \underline{transverse} part

Returning to Ampere's law, we see that the ten

\[
\nabla \left( \frac{\partial \phi}{\partial t} \right) = -\nabla \int d^3r' \left( \frac{\nabla' \cdot \vec{J}(r',t)}{|r-r'|} \right) = 4\pi \frac{\partial}{\partial t} \vec{J}_{\parallel}(\vec{r},t)
\]

So, Ampere's law becomes

\[
\nabla \cdot \vec{A} = \frac{4\pi}{c} \frac{\partial}{\partial t} J - \frac{4\pi}{c} \frac{\partial}{\partial t} \vec{J}_{\parallel}
\]

\[
\nabla \cdot \vec{A} = \frac{4\pi}{c} \frac{\partial}{\partial t} J_{\parallel}
\]

In Coulomb gauge, only the transverse part of \( J \)
serves as a source for \( \vec{A} \).

\( \vec{A} \) describes the \underline{transverse modes}, i.e.
the \underline{EM radiation} (recall in EM waves, the fields are always \( \perp \) direction of propagation).

\( \phi \) describes the \underline{longitudinal modes}

Coulomb gauge is not Lorentz invariant - if \( \nabla \cdot \vec{A} = 0 \)
in one inertial reference frame, in general \( \nabla \cdot \vec{A} \neq 0 \) in another.

In Coulomb gauge, if \( g = 0 \), then \( \phi = 0 \) and

\[
\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t}
\]
Transverse + Longitudinal Parts of vector functions

To prove the preceding claim, \( \vec{f} = \vec{f}_n + \vec{f}_t \),
where \( \nabla \times \vec{f}_n = 0 \) and \( \nabla \cdot \vec{f}_t = 0 \), we first dispose of
proving Helmholtz Theorem.

Helmholtz Theorem: For a vector function \( \vec{f}(\vec{r}) \) if one
knows the divergence and curl of \( \vec{f} \) then one
uniquely determine \( \vec{f} \) itself.

That is, if

\[
\nabla \cdot \vec{f} = 4\pi D(\vec{r}) \quad \text{where} \quad D(\vec{r}) \quad \text{is a known}
\]
scalar function

\[
\nabla \times \vec{f} = 4\pi \vec{C}(\vec{r}) \quad \text{where} \quad \vec{C}(\vec{r}) \quad \text{is a known}
\]
vector function

Then one can solve for

And it will defined boundary conditions on \( f \)
are known (here we will assume \( f(\vec{r}) \rightarrow 0 \) as \( \vec{r} \rightarrow \infty \))
Then there is a unique solution for \( f(\vec{r}) \).

we prove this by construction!

Assume a solution of the form

\[
\vec{f} = -\nabla \phi + \nabla \times \vec{W} \quad \text{where} \quad \phi \quad \text{a scalar}
\]
and \( \vec{W} \quad \text{a vector} \)

Now we show that we can find such a solution.
Further consider
\[ \mathbf{\nabla} \cdot \mathbf{\tilde{f}} = -\nabla^2 \varphi + \mathbf{\nabla} \cdot (\mathbf{\nabla} \times \mathbf{\tilde{W}}) = -\nabla^2 \varphi + \mathbf{0} = 4\pi D(\mathbf{r}) \]

So \[ -\nabla^2 \varphi = 4\pi D(\mathbf{r}) \] This is just Poisson's equation, we saw in electrostatics.

Solution when \( \varphi(\mathbf{r}) \to 0 \) as \( r \to \infty \) is given by
\[ \varphi(\mathbf{r}) = \frac{\int d^3r'}{4\pi |r-r'|} \cdot \quad \text{Coulomb-like integral solution} \]

Now consider
\[ \mathbf{\nabla} \times \mathbf{\tilde{f}} = -\mathbf{\nabla} \times \mathbf{\nabla} \varphi + \mathbf{\nabla} \times (\mathbf{\nabla} \times \mathbf{\tilde{W}}) = \mathbf{0} - \nabla \tilde{W} + \mathbf{\nabla} \times (\mathbf{\nabla} \times \mathbf{\tilde{W}}) = 4\pi \tilde{C}(\mathbf{r}) \]

Choose a gauge in which \( \mathbf{\nabla} \cdot \mathbf{\tilde{W}} = 0 \) (just like Coulomb gauge in magnetostatics).

Then \[ -\nabla^2 \tilde{W} = 4\pi \tilde{C}(\mathbf{r}) \]

\[ \tilde{W}(\mathbf{r}) = \frac{\int d^3r'}{4\pi |r-r'|} \cdot \quad \text{just like solution for vector potential in magnetostatics} \]

So we have constructed a solution
\[ \tilde{f}(\mathbf{r}) = -\mathbf{\nabla} \varphi + \mathbf{\nabla} \times \mathbf{\tilde{W}} = -\mathbf{\nabla} \left( \frac{\int d^3r'}{4\pi |r-r'|} \cdot \right) + \mathbf{\nabla} \times \left( \frac{\int d^3r'}{4\pi |r-r'|} \cdot \right) \]

Where \( \mathbf{\nabla} \cdot \mathbf{\tilde{f}} = 4\pi D \) and \( \mathbf{\nabla} \times \mathbf{\tilde{f}} = 4\pi \tilde{C} \)
Note: For above solution to be well defined, the integrals must converge. They will converge if the "sources" \( \text{D}(\mathbf{r}) \) ad \( \bar{\text{C}}(\mathbf{r}) \) are sufficiently "localized" in space, i.e. \( \text{D}(\mathbf{r}) \to 0 \), \( \bar{\text{C}}(\mathbf{r}) \to 0 \) sufficiently fast as \( \mathbf{r} \to \infty \).

Now we show that the above solution is unique.

Suppose there was another solution \( \tilde{f} \) such that

\[
\nabla \cdot \tilde{f} = 4\pi D \quad \text{and} \quad \nabla \times \tilde{f} = 4\pi C
\]

Consider \( \tilde{h} = \tilde{f} - \bar{f} \) then

\[
\nabla \cdot \tilde{h} = 0 \quad \text{and} \quad \nabla \times \tilde{h} = 0
\]

Can show that only such \( \tilde{h} \) that also has \( \tilde{h}(\mathbf{r}) \to 0 \) as \( \mathbf{r} \to \infty \) is \( \tilde{h} \equiv 0 \), so \( \bar{f} = \tilde{f} \)

and solution is unique.

As a consequence of Helmholtz Theorem, we have also shown the following:

1. Any vector function \( \tilde{f} \) can be written as a sum of a scalar and vector potential

\[
\tilde{f} = -\nabla \phi + \nabla \times \mathbf{W}
\]

or equivalently
Any vector function \( \vec{F} \) can be written in terms of a curl free and a divergenceless part:

\[
\vec{F} = \vec{F}_\parallel + \vec{F}_\perp \quad \text{where} \quad \nabla \times \vec{F}_\parallel = 0 \quad \text{curlfree} \quad \nabla \cdot \vec{F}_\perp = 0 \quad \text{divergenceless}
\]

where

\[
\begin{align*}
\vec{F}_\parallel (\vec{r}) &= -\nabla \Phi (\vec{r}) = -\nabla \int \frac{d^3 r'}{4\pi} \frac{\Phi (\vec{r}')}{|\vec{r} - \vec{r}'|} \\
\vec{F}_\perp (\vec{r}) &= \nabla \times \vec{W} (\vec{r}) = \nabla \times \int \frac{d^3 r'}{4\pi} \frac{\nabla' \times \vec{F} (\vec{r}')}{|\vec{r} - \vec{r}'|}
\end{align*}
\]

where \( \Phi \) above we used \( \Phi (\vec{r}') = \frac{1}{4\pi} \frac{\nabla'}{\vec{r}'} \cdot \vec{F} (\vec{r}') \)

\( \vec{W} (\vec{r}) = \frac{1}{4\pi} \frac{\nabla \times \vec{F} (\vec{r})}{\vec{r}} \)

\( \vec{F}_\parallel \) is called the **longitudinal** part of \( \vec{F} \)

\( \vec{F}_\perp \) is called the **transverse** part of \( \vec{F} \)

To understand the reason for these names, we need to consider the Fourier transforms.

Above can be generalized to situations where \( \vec{F} \) satisfies other boundary conditions, specifically when \( \vec{F} \) has a specified value on a given boundary surface. One must replace \( \frac{1}{|\vec{r} - \vec{r}'|} \) by the appropriate Greens function — see more to come!
Discussion regarding Fourier transforms

\[ \hat{f}(\vec{r}) = \frac{1}{(2\pi)^3} \int d^3k \ e^{-i\vec{k} \cdot \vec{r}} \ f(\vec{k}) \quad \text{Fourier transform} \]

\[ \hat{f}(\vec{r}) = \int d^3r \ e^{-i\vec{k} \cdot \vec{r}} \ f(\vec{r}) \quad \text{inverse transform} \]

Some special cases well worth remembering

1. Transform of Dirac function

\[ \int d^3r \ e^{-i\vec{k} \cdot \vec{r}} \delta(\vec{r} - \vec{r}_0) = e^{-i\vec{k} \cdot \vec{r}_0} \]

\[ \Rightarrow \delta(\vec{r} - \vec{r}_0) = \int d^3k \ e^{+i\vec{k} \cdot \vec{r}} \ e^{-i\vec{k} \cdot \vec{r}_0} \]

\[ \delta(\vec{r} - \vec{r}_0) = \int d^3k \ e^{i\vec{k} \cdot (\vec{r} - \vec{r}_0)} \]

or letting \( \vec{r} \leftrightarrow \vec{r}_0 \) with the above

\[ \delta(\vec{r} - \vec{r}_0) = \int d^3k \ e^{i\vec{k} \cdot (\vec{r} - \vec{r}_0)} \]

2. Transform of Coulomb potential \( \frac{1}{|\vec{r} - \vec{r}'|} \)

We know

\[ \nabla^2 \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) = -4\pi \delta(\vec{r} - \vec{r}') \]

Suppose \( f(\vec{r}) = \int d^3r \ e^{-i\vec{k} \cdot \vec{r}} \frac{1}{|\vec{r} - \vec{r}'|} \)

is the Fourier transform of \( \frac{1}{|\vec{r} - \vec{r}'|} \)
Substitute \( \left\{ \begin{array}{l} \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{k}) \\ \delta(\mathbf{r}-\mathbf{r}') = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \end{array} \right. \)

into above Poisson equation

\[ \nabla^2 \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r}) = -4\pi \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \]

 operates only on \( \mathbf{r} \)

so move inside integral

\[ \nabla^2 e^{i\mathbf{k}\cdot\mathbf{r}} = \nabla \cdot (\nabla e^{i\mathbf{k}\cdot\mathbf{r}}) \]

\[ \begin{align*}
\nabla e^{i\mathbf{k}\cdot\mathbf{r}} &= \sum_{z=1}^{3} \hat{x}_z \frac{\partial}{\partial x_z} e^{i\mathbf{k}\cdot\mathbf{r}} \\
&= \sum_{z=1}^{3} \hat{x}_z ik_z e^{i\mathbf{k}\cdot\mathbf{r}} \\
&= \mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} \\
\text{where } \hat{x}_1, \hat{x}_2, \hat{x}_3 &= \hat{x}, \hat{y}, \hat{z} \\
\nabla \cdot (i\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}}) &= (i\mathbf{k}) \cdot (i\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} = -k^2 e^{i\mathbf{k}\cdot\mathbf{r}} \\
\n\nabla^2 e^{i\mathbf{k}\cdot\mathbf{r}} &= -k^2 e^{i\mathbf{k}\cdot\mathbf{r}}
\end{align*} \]

Poisson equation gives

\[ \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} (-k^2) f(\mathbf{k}) = -4\pi \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\mathbf{k}\cdot\mathbf{r}'} \]

\[ \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} \left[ -k^2 f(\mathbf{k}) \right] = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} \left[ -4\pi e^{-i\mathbf{k}\cdot\mathbf{r}'} \right] \]
As is true for Fourier series, so it is true for Fourier transforms: If two functions are equal, then their Fourier transforms are equal.

\[ \Rightarrow -k^2 f(k) = -\frac{4\pi}{k} e^{-ik \cdot \vec{r}} \]

\[ f(k) = \frac{4\pi}{k^2} e^{-\frac{1}{2}k \cdot \vec{r}} \]

\[ \Rightarrow \text{the Fourier transform of} \frac{1}{r^2 - \vec{r} ' } \]