Electrostatics

\[-\nabla^2 \phi = \rho \text{ for } \vec{E} = -\nabla \phi \text{ (statics only)}\]

Physical meaning of the potential \(\phi\)

Work done to move a test charge \(q\) from \(\vec{r}_1\) to \(\vec{r}_2\) in presence of an electric field \(\vec{E}\) is

\[W_{12} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{l}\]

where \(\vec{F}\) is the force required to move the charge.

Since \(\vec{E}\) exerts a force \(q\vec{E}\) on the charge, \(\vec{F}\) must counterbalance this electric force so we can move the charge quasi-statically \(\Rightarrow \vec{F} = -q\vec{E}\)

\[W_{12} = -q\int_{\vec{r}_1}^{\vec{r}_2} d\vec{l} \cdot \vec{E} = q\int_{\vec{r}_1}^{\vec{r}_2} d\vec{l} \cdot \nabla \phi = q \left[ \phi(\vec{r}_2) - \phi(\vec{r}_1) \right]\]

\[\phi(\vec{r}_2) - \phi(\vec{r}_1) = \frac{W_{12}}{q}\]

Difference in potential between two points is the work per unit charge to move a test charge between the two points.

Only true in statics because \(\vec{E} = -\nabla \phi\) only in statics
Green's Functions - part I

\[- \nabla^2 \phi = 4\pi f\]

We already know that for a point charge \( q \) at position \( \vec{r}' \),
\( f(\vec{r}) = q \delta(\vec{r} - \vec{r}') \),
the solution to the above is:

\[\phi(\vec{r}) = \frac{q}{4\pi |\vec{r} - \vec{r}'|}\]
\[\Rightarrow -\nabla^2 \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) = 4\pi \delta(\vec{r} - \vec{r}')\]

We call the special solution for a point source
the Green function for the differential operator

\[-\nabla^2 G(\vec{r},\vec{r}') = 4\pi \delta(\vec{r} - \vec{r}')\]

\( G(\vec{r},\vec{r}') \) gives the potential at position \( \vec{r} \) due
to a unit source at position \( \vec{r}' \).

Generally, one also has to specify a desired
boundary condition for the Green function on
the boundary of the system.

For the Coulomb solution for a point charge
the implicit boundary condition is that the
potential vanish infinitely far from the charge.

\[G(\vec{r},\vec{r}') \to 0 \quad \text{as} \quad |\vec{r} - \vec{r}'| \to \infty\]

boundary of the system is taken to infinity
If one knows the Green's function, then one can find the solution for any distribution of sources $p(\vec{r'})$

$$\phi(\vec{r}) = \int d^3r' G(\vec{r}, \vec{r}') p(\vec{r}')$$

**proof:**

$$-\nabla^2 \phi = \int d^3r' \left[-\nabla^2 G(\vec{r}, \vec{r}')\right] p(\vec{r}')$$

$$= \int d^3r' \left[4\pi \delta(\vec{r} - \vec{r}')\right] p(\vec{r}')$$

$$= 4\pi p(\vec{r})$$

We will return to concept of Green's function when we discuss solution of Poisson's equation in a finite volume.

We will also see Green's functions again when we discuss solution of the homogeneous wave equation.
The Coulomb problem as a boundary value problem

Consider a conducting sphere of radius \( R \) with net charge \( q \) (as \( R \to 0 \) we get a point charge).

What is \( \phi(r) \)? What is \( E(r) \)?

Review: Properties of conductors in electrostatics

1) \( \vec{E} = 0 \) inside conductor - if \( \vec{E} \neq 0 \) then a current \( \vec{j} = \sigma \vec{E} \) flows and it is not statics (\( \sigma \) is conductivity)
2) \( j = 0 \) inside conductor - if \( \vec{E} = 0 \) inside, then \( \vec{\nabla} \cdot \vec{E} = \nabla \cdot \nabla \phi = 0 \)
3) Any net charge on the conductor must lie on the surface - follows from (2)
4) \( \phi \) constant throughout conductor - if \( \vec{E} = 0 \)
   then \( \vec{E} = \vec{\nabla} \phi \Rightarrow \phi \) is constant
5) Just outside the conductor, \( \vec{E} \) is \( \perp \) to surface.
   - If \( \vec{E} \) has a component \( \parallel \) to surface then it exerts a force on electrons at the surface
     leading to a surface current - so would not be static

For conducting sphere, \( j = 0 \) for \( r > R \) and \( r < R \)
all charge on the surface \( \Rightarrow \nabla^2 \phi = 0 \) for \( r < R \)

spherical symmetry \( \Rightarrow \) expect spherically symmetric solution

\( \Rightarrow \phi(r) \) depends only on \( r = |\vec{r}| \)
Solve Laplace's equation by writing $\nabla^2 \phi$ in spherical coordinates.

Only the radial terms do not vanish.

$$\nabla^2 \phi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = 0$$

$$r^2 \frac{d\phi}{dr} = -C_0 \quad \text{a constant}$$

$$\frac{d\phi}{dr} = -\frac{C_0}{r^2}$$

$$\phi(r) = \frac{C_0}{r} + C_1, \quad C_1 \quad \text{a constant}$$

"outside" \quad r > R \quad \phi_{\text{out}}(r) = \frac{C_{0\text{out}}}{r} + C_{1\text{out}}

"inside" \quad r < R \quad \phi_{\text{in}}(r) = \frac{C_{0\text{in}}}{r} + C_{1\text{in}}

The solution "outside" does not necessarily go smoothly into the solution "inside" because of the charge layer at $r = R$ that separates the two regions. We need to determine the constants $C_{0\text{in}}, C_{0\text{out}}, C_{1\text{in}}, C_{1\text{out}}$ by applying boundary conditions corresponding to the physical situation.

1. For $r > R$, assume $\phi \to 0$ as $r \to \infty$ - boundary condition at infinity

   $$\Rightarrow C_{1\text{out}} = 0$$

   $$\phi_{\text{out}}(r) = \frac{C_{0\text{out}}}{r}$$

   recover the expected Coulomb form.
2) For $r < R$

i) We could use the fact that the region $r < R$ is a conductor with $\phi = \text{constant}$ to conclude $C_{in}^m = 0$

ii) or, if we were dealing with a charged shell instead of a conductor, we could argue as follows:
no charge at origin $r=0 \Rightarrow$ expect $\phi$ should be finite at origin $\Rightarrow C_{in}^m = 0$

So $\phi_{in}^m (r) = C_{in} \text{ a constant}$

3) Now we need boundary conditions at $r=R$ where "inside" and "outside" meet.

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Review: Electric field and potential at a surface charge layer

A general surface $S$ with surface charge density $\sigma(r)$ for $r$ on $S$. $\sigma(r) da$ is total charge in area $da$ on surface.

i) Take "Gaussian pillbox" surface about point $\vec{r}$ on the surface $S'$

- top and bottom areas of pillbox $da$,
- side view of pillbox $dl$.

Gauss' law in integral form $\int_S \sigma \hat{r} \cdot \vec{E} = 4\pi Q_{\text{enclosed}}$
Expect \( \vec{E} \) is finite \( \rightarrow \) contribution from sides of pillbox vanish as \( dl \to 0 \).

\[
\oint \vec{d}a \cdot \vec{E} = \int_{\text{top}} \vec{d}a \cdot \vec{E} + \int_{\text{bottom}} \vec{d}a \cdot \vec{E}
\]

Since \( \vec{d}a \) is small

\( \vec{E} \) \text{ top } is electric field at \( \vec{r} \) just above the surface \( S \)

\( \vec{E} \) \text{ bottom } is electric field at \( \vec{r} \) just below the surface \( S \)

\( \hat{n} \) \text{ top } = \hat{n} \text{ is outward normal on top}

\( \hat{n} \) \text{ bottom } = -\hat{n} \text{ is outward normal on bottom}

\[
= \left( \vec{E} \text{ top } - \vec{E} \text{ bottom } \right) \cdot \hat{n} \ \text{d}a = 4\pi \sigma \text{ enclosed } = 4\pi \sigma(\vec{r}) \ \text{d}a
\]

\[
\begin{align*}
\left( \vec{E} \text{ top } - \vec{E} \text{ bottom } \right) \cdot \hat{n} = 4\pi \sigma(\vec{r}) \quad \text{discontinuity in normal component of } \vec{E}
\end{align*}
\]

ii) Take "American loop" \( C \) at surface about point \( \vec{r} \).

\[
\nabla \times \vec{E} = 0 \quad \Rightarrow \quad \oint \vec{d}l \cdot \vec{E} = 0 \quad \text{since } \vec{E} \text{ is finite at surface, if take sides } \vec{d}l' \to 0 \text{ their contribution to integral vanishes}
\]

\[
\Rightarrow \oint \vec{d}l \cdot \vec{E} = \left( \vec{E} \text{ top } - \vec{E} \text{ bottom } \right) \cdot \vec{d}l = 0
\]

where \( \vec{d}l \) is any infinitesimal tangent to the surface at \( \vec{r} \).
\[ \nabla \phi \Rightarrow \phi (r_2) - \phi (r_1) = -\oint_{r_1} \vec{E} \cdot d\vec{r} \]

Take \( r_2 \) just above \( r \) on surface \( \oint \vec{E} \cdot d\vec{r} \to 0 \)

Take \( r_1 \) just below \( r \) on surface \( \oint \vec{E} \cdot d\vec{r} \to 0 \)

Since \( \vec{E} \) is finite \( \Rightarrow \oint \vec{E} \cdot d\vec{r} \to 0 \)

\[ \Rightarrow \phi_{\text{top}} = \phi_{\text{bottom}} \]

potential \( \phi \) is continuous at surface charge layer

Can rewrite (i) as

\[ \left( -\nabla \phi_{\text{top}} + \nabla \phi_{\text{bottom}} \right) \cdot \hat{M} = 4\pi \sigma \]

\[ -\frac{\partial \phi_{\text{top}}}{\partial M} + \frac{\partial \phi_{\text{bottom}}}{\partial M} = 4\pi \sigma \]

C directional derivative of \( \phi \) in direction \( \hat{M} \)

do not consider derivative of \( \phi \) at surface

Apply to conducting sphere

\[ \phi \text{ continuous} \Rightarrow \phi_{\text{in}}(R) = \phi_{\text{out}}(R) \]

\[ c_{\text{in}} = c_{\text{out}} \frac{1}{R} \]

Only one unknown left
normal derivative of $\phi$ is discontinuous

\[ \frac{\partial \phi_{\text{top}}}{\partial n} + \frac{\partial \phi_{\text{bottom}}}{\partial n} = 4\pi \sigma \]

where $\hat{n}$ is the radial direction.

\[ \left[ -\frac{d\phi^{\text{out}}}{dr} + \frac{d\phi^{\text{in}}}{dr} \right]_{r=R} = 4\pi \sigma \]

but $\frac{d\phi^{\text{in}}}{dr} = 0$ as $\phi^{\text{in}} = \text{constant}$

\[ \left. -\frac{d\phi^{\text{out}}}{dr} \right|_{r=R} = 4\pi \sigma \]

charge $Q$ is uniformly distributed on surface at $R$

\[ -\frac{d}{dr} \left( \frac{C_0^{\text{out}}}{r} \right)_{r=R} = \frac{C_0^{\text{out}}}{R^2} = 4\pi \sigma = 4\pi \left( \frac{Q}{4\pi R^2} \right) = \frac{Q}{R^2} \]

\[ \Rightarrow C_0^{\text{out}} = \frac{Q}{R}, \quad C_1^{\text{in}} = \frac{e_0^{\text{out}}}{e} = \frac{Q}{e R} \]

\[ \phi(r) = \begin{cases} \frac{Q}{R} & r < R \text{ inside} \\ \frac{Q}{r} & r > R \text{ outside} \end{cases} \]

\[ \Rightarrow \mathbf{E} = -\nabla \phi = \left. -\frac{d\phi}{dr} \right| = \begin{cases} 0 & r < R \text{ inside} \\ \frac{Q}{r^2} & r > R \text{ outside} \end{cases} \]

we get familiar Coulomb solution!
Summary: We can view the preceding solution for \( \phi_{\text{out}} \) as solving Laplace's equation \( \nabla^2 \phi = 0 \) subject to a specified boundary condition on the normal derivative of \( \phi \) at the boundary \( r = R \) of the "outside" region of the system.

Alternate problem:

Another physical situation would be to connect a conducting sphere to a battery that charges the sphere to a fixed voltage \( \phi_0 \) (stat volts!) with respect to ground \( \phi = 0 \) at \( r = \infty \).

As before, outside the sphere \( \phi = \frac{C_0}{r} \).

Now the boundary condition is to specify the value of \( \phi \) on the boundary of the outside region, i.e.

\[
\phi(R) = \phi_0
\]

\[
\Rightarrow \quad \frac{C_0}{R} = \phi_0, \quad C_0 = \phi_0 R
\]

\[
\phi(r) = \phi_0 \frac{R}{r}
\]

(from preceding solution, we know that charging the sphere to voltage \( \phi_0 \) (stat volts) induces a net charge \( q = \phi_0 R \) on it.)
These two versions of the conducting sphere problem are examples of a more general boundary value problem.

Solve $\nabla^2 \phi = 0$ in a given region of space subject to one of the following two types of boundary conditions on the boundary surfaces of the region.

i) Neumann boundary condition

$\frac{\partial \phi}{\partial n} = \text{normal derivative of } \phi \text{ is specified on the boundary surface.}$

ii) Dirichlet boundary condition

$\phi = \text{value of } \phi \text{ is specified on the boundary surfaces.}$

If the boundary surfaces consist of disjoint pieces, it is possible to specify either (i) or (ii) on each piece separately to get a mixed boundary value problem.
Some more problems

Infinite conducting wire of radius $R$ with line charge density $\lambda = \text{charge per unit length}$.

Surface charge $\sigma = \frac{\lambda}{2\pi R}$.

Expect cylindrical symmetry, $\phi$ depends only on cylindrical coord $r$.

$\nabla^2 \phi = 0$ for $r > R$, $r < R$.

Use $\nabla^2$ in cylindrical coords - only radial term.

"$r$" is cylindrical radial coordinate.

"$\nabla^2 \phi = \frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) = 0$"

$r \frac{d\phi}{dr} = C_0$ constant

$\frac{d\phi}{dr} = \frac{C_0}{r}$

$\phi(r) = C_0 \ln r + C_1$, const

Note: one cannot now choose $\phi \to 0$ as $r \to \infty$!

One needs to fix zero of $\phi$ at some other radius, a convenient choice is $r = R$, but any other choice could also be made.
\[ \phi_{\text{out}} = C_0 \ln r + C_{1,\text{out}} \]
\[ \phi_{\text{in}} = C_0 \ln r + C_{1,\text{in}} \]

\[ \phi_{\text{in}} = \text{const}\, \text{in conductor} \Rightarrow C_{0,\text{in}} = 0 \]

or \( \phi_{\text{in}} \) should not diverge as \( r \to 0 \Rightarrow C_{0,\text{in}} = 0 \)

So \( \phi_{\text{in}} = C_{1,\text{in}} \, \text{constant} \)

Boundary condition at \( r = R \):

\[ \left[ -\frac{d\phi_{\text{out}}}{dr} + \frac{d\phi_{\text{in}}}{dr} \right]_{r = R} = 4\pi \sigma \]

\[ -\frac{C_{0,\text{out}}}{R} = 4\pi \sigma = 4\pi \left( \frac{\lambda}{2\pi R} \right) = \frac{2\lambda}{R} \]

\[ C_{0,\text{out}} = -2\lambda \]

\[ \phi_{\text{out}}(R) = -2\lambda \ln R + C_{1,\text{out}} \]

Continuity of \( \phi \):

\[ \phi_{\text{in}}(R) = \phi_{\text{out}}(R) \Rightarrow C_{1,\text{in}} = -2\lambda \ln R + C_{1,\text{out}} \]

Remaining const \( C_{1,\text{out}} \) is not too important as it is just a common additive constant to both \( \phi_{\text{in}} \) and \( \phi_{\text{out}} \) does not change \( \mathbf{\mathcal{E}} = -\nabla \phi \).

If we use the condition \( \phi(R) = 0 \) then we can solve for \( C_{1,\text{out}} \),
\[ 0 = -2\lambda \ln R + C_1^{\text{out}} \Rightarrow C_1^{\text{out}} = 2\lambda \ln R \]

\[ \Rightarrow \phi(r) = \begin{cases} -2\lambda \ln(r/R) & r > R \\ 0 & r < R \end{cases} \]

\( \vec{E}(r) = \begin{cases} \frac{2\lambda}{r} & r > R \\ 0 & r < R \end{cases} \)

- infinite conducting half space
- \( \sigma \) uniform surface charge density
- conductor
- expect \( \phi \) depends only on \( x \)

\[ \nabla^2 \phi = \frac{d^2 \phi}{dx^2} = 0 \]

\[ \Rightarrow \begin{cases} \phi^+(x) = C_0^+ x + C_1^+ & x > 0 \\ \phi^-(x) = C_0^- x + C_1^- & x < 0 \end{cases} \]

for \( x < 0 \), \( \phi = \text{const at conductor} \Rightarrow C_0^- = 0 \)

at \( x = 0 \), \( \phi \) continuous \( \Rightarrow \phi^-(0) = \phi^+(0) \)

\[ C_1^- = C_1^+ \]

\[ \frac{d\phi}{dx} \text{ discontinuous} \Rightarrow -\frac{d\phi^+}{dx} \bigg|_{x=0} = 4\pi \sigma \]

\[ C_0^+ = -4\pi \sigma \]

\[ \Rightarrow \phi(x) = \begin{cases} -4\pi \sigma x + C_1^+ & x > 0 \\ C_1^+ & x < 0 \end{cases} \]

const \( C_1^+ \) does not change value of \( \vec{E} \)
as for the wire, we cannot choose \( \phi \to 0 \) as \( x \to \infty \). We can set \( \phi = \text{const} \). If we choose \( \phi = 0 \) at \( x = 0 \), then \( c_1^+ = 0 \).

\[
-\nabla \phi = E = \begin{cases} 
4\pi \sigma & x > 0 \\
0 & x < 0 
\end{cases}
\]

**Infinite charged plane**

Similar to previous problem but now no conductor at \( x < 0 \), just free space on both sides of the charged plane at \( x = 0 \).

\[ \nabla^2 \phi = \frac{d^2 \phi}{dx^2} = 0 \implies \phi^+ = c_0^+ x + c_1^+ \quad x > 0 \]
\[ \phi^- = c_0^- x + c_1^- \quad x < 0 \]

**Continuity of \( \phi \) at \( x = 0 \)**

\[ \phi^+(0) = \phi^-(0) \implies c_1^+ = c_1^- \]

**Discontinuity of \( \frac{d\phi}{dx} \) at \( x = 0 \)**

\[ -\frac{d\phi^+}{dx} + \frac{d\phi^-}{dx} = 4\pi \sigma \]
\[ -c_0^+ + c_0^- = 4\pi \sigma \]

Define \( \varepsilon_0 = \frac{c_0^+ + c_0^-}{2} \)

Then we can write
\[ C_0^- = \bar{C}_0 + 2\pi \sigma \]
\[ C_0^+ = \bar{C}_0 - 2\pi \sigma \]

\[ \phi = \begin{cases} 
2\pi \sigma x + \bar{C}_0 x + C_1^+ & x > 0 \\
-2\pi \sigma x + \bar{C}_0 x + C_1^- & x < 0 
\end{cases} \]

\[ -\frac{d\phi}{dx} = \vec{E} = \begin{cases} 
(2\pi \sigma - \bar{C}_0) \hat{x} & x > 0 \\
(-2\pi \sigma - \bar{C}_0) \hat{x} & x < 0 
\end{cases} \]

Constant \( C_1^+ \) does not affect \( \vec{E} \) — additive const to \( \phi \).
\( \bar{C}_0 \) represents constant uniform electric field \(-\bar{C}_0 \hat{x}\), that exists independently of the charged surface, i.e. remains even as \( \sigma \to 0 \).

If we assumed that all \( \vec{E} \) fields are just those arising from the plane, then we can set \( \bar{C}_0 = 0 \).
Equivalently, if the plane is the only source of \( \vec{E} \), then we expect \( \phi \) depends only on \( |x| \) by symmetry.
\[ \Rightarrow C_0^- = -C_0^+ \] and again \( \bar{C}_0 = 0 \). In this case

\[ \phi(x) = \begin{cases} 
-2\pi \sigma x & x > 0 \\
2\pi \sigma x & x < 0 
\end{cases} \]

we also set \( C_1^+ = 0 \) here, corresponding to \( \phi(0) = 0 \).

\[ \vec{E}(x) = \begin{cases} 
2\pi \sigma \hat{x} & x > 0 \\
-2\pi \sigma \hat{x} & x < 0 
\end{cases} \]

\( \vec{E} \) is constant but oppositely directed on either side of the charged plane.