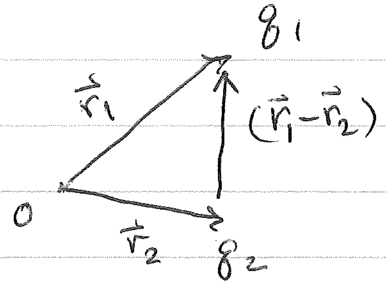
Example two charges $q_1$ at $\vec{r}_1$ and $q_2$ at $\vec{r}_2$

$$q_1 + q_2 = q \neq 0$$




monopole $q_1 + q_2 = q$

dipole $\vec{p} = q_1 \vec{r}_1 + q_2 \vec{r}_2$

quadrupole $\overleftrightarrow{Q} = (3\vec{r}_1\vec{r}_1 - r_1^2 \overleftrightarrow{I})q_1 + (3\vec{r}_2\vec{r}_2 - r_2^2 \overleftrightarrow{I})q_2$

We can make the dipole moment vanish by shifting to a new coord system $\vec{r}\,' = \vec{r} - \vec{d}$ where $\vec{d} = \dfrac{\vec{p}}{q}$

$$\vec{r}\,' = \vec{r} - \frac{q_1\vec{r}_1 + q_2\vec{r}_2}{q_1 + q_2} = \frac{q_1(\vec{r} - \vec{r}_1) + q_2(\vec{r} - \vec{r}_2)}{q_1 + q_2}$$

positions of $q_1, q_2$ in new coords are

$$\vec{r}_1\,' = \frac{q_2}{q_1 + q_2}(\vec{r}_1 - \vec{r}_2)$$

$$\vec{r}_2\,' = \frac{-q_1}{q_1 + q_2}(\vec{r}_1 - \vec{r}_2)$$

Origin of new coord system is at

lies along vector from $\vec{r}_2$ to $\vec{r}_1$

$$\vec{r}\,' = 0 \Rightarrow \vec{r} = \frac{q_1\vec{r}_1 + q_2\vec{r}_2}{q_1 + q_2}$$ "center of charge"

for many charges $q_i$ at positions $\vec{r}_i$, the origin that makes dipole moment vanish is at

$$\vec{r} = \frac{\sum_i q_i \vec{r}_i}{\sum_i q_i}$$

In this coord system

$$\vec{p}\,' = q_1 \vec{r}_1\,' + q_2 \vec{r}_2\,' = \frac{q_1 q_2}{q_1+q_2}(\vec{r}_1-\vec{r}_2) - \frac{q_2 q_1}{q_1+q_2}(\vec{r}_1-\vec{r}_2)$$

$$= 0 \quad \text{as it must be!}$$

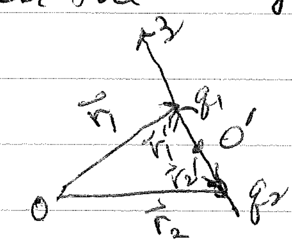
<u>Quadrupole moment</u> in the coord system in which $\vec{p}\,'=0$ the quadrupole tensor is

$$\overleftrightarrow{Q}' = \left[3\vec{r}_1\,'\vec{r}_1\,' - (r_1')^2\overleftrightarrow{I}\right]q_1 + \left[3\vec{r}_2\,'\vec{r}_2\,' - (r_2')^2\overleftrightarrow{I}\right]q_2$$

let us choose ~~coord~~ spherical coordinates with origin at $O'$ and $\hat{z}$ axis aligned along $\vec{r}_1-\vec{r}_2$, so that

$$\vec{r}_1-\vec{r}_2 = s\hat{z} \quad \text{where } s = |\vec{r}_1-\vec{r}_2| \text{ is separation between the charges}$$

then
$$\vec{r}_1\,' = \frac{q_2}{q_1+q_2}s\hat{z}$$
$$\vec{r}_2\,' = \frac{-q_1}{q_1+q_2}s\hat{z}$$




$$\overleftrightarrow{Q}' = \left(\frac{q_2}{q_1+q_2}\right)^2 q_1\left[3s^2\hat{z}\hat{z} - s^2\overleftrightarrow{I}\right]$$
$$+ \left(\frac{-q_1}{q_1+q_2}\right)^2 q_2\left[3s^2\hat{z}\hat{z} - s^2\overleftrightarrow{I}\right]$$

$$\overleftrightarrow{Q}' = \frac{q_2^2 q_1 + q_1^2 q_2}{(q_1+q_2)^2} s^2 \left[ 3\hat{z}\hat{z} - \overleftrightarrow{I} \right]$$

$$= \frac{q_1 q_2}{q_1+q_2} s^2 \left[ 3\hat{z}\hat{z} - \overleftrightarrow{I} \right]$$

$$Q'_{ij} = \frac{q_1 q_2}{q_1+q_2} s^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

in $xyz$ coord system

as $\hat{z}\hat{z} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\overleftrightarrow{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(check: $\overleftrightarrow{Q}'$ is traceless - $-1-1+2=0$)

the contribution of quadrupole to the potential is

$$\phi_{quad} = \frac{1}{2} \frac{\hat{r} \cdot \overleftrightarrow{Q} \cdot \hat{r}}{r^3}$$

$\hat{r} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$ in $xyz$ coords

with origin at $O'$ this becomes

$$\phi_{quad} = \frac{s^2}{2r^3} \frac{q_1 q_2}{q_1+q_2} (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

do matrix multiplications

$$\phi_{quad} = \frac{s^2}{2r^3} \frac{q_1 q_2}{q_1+q_2} (2\cos^2\theta - \sin^2\theta)$$

independent of $\varphi$ as it must be due to azimuthal symmetry

$$\Phi_{quad} = \frac{s^2}{2r^3}\frac{q_1 q_2}{q_1+q_2}(2\cos^2\theta - \sin^2\theta)$$

use $\sin^2\theta = 1-\cos^2\theta \Rightarrow 2\cos^2\theta - \sin^2\theta = 3\cos^2\theta - 1$

use $\cos^2\theta = \frac{1+\cos 2\theta}{2} \Rightarrow 3\cos^2\theta - 1 = \frac{1+3\cos 2\theta}{2}$

So $\Phi_{quad} = \frac{s^2}{2r^3}\frac{q_1 q_2}{q_1+q_2}\frac{1+3\cos 2\theta}{2}$

compare to

$$\Phi_{dipole} = \frac{p\cos\theta}{r^2}$$

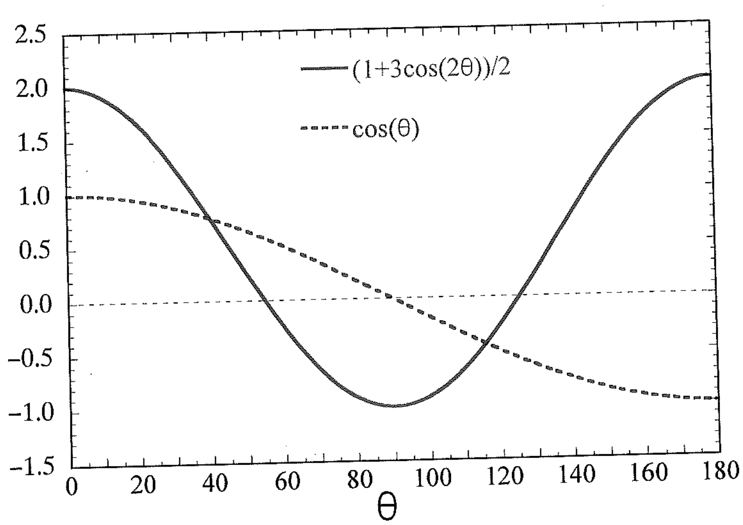
Note, if we average over $\theta$ then

$$\int_0^\pi d\theta \sin\theta \, \Phi_{quad} \propto \int_0^\pi d\theta \sin\theta\,(3\cos^2\theta - 1) = \left[-\cos^3\theta + \cos\theta\right]_0^\pi$$

$$= 0$$

similarly $\int_0^\pi d\theta \sin\theta \, \Phi_{dipole} \propto \int_0^\pi d\theta \sin\theta \cos\theta = \left[-\frac{1}{2}\cos^2\theta\right]_0^\pi$

$$= 0$$



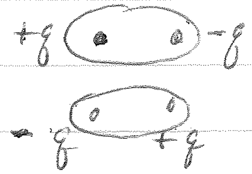
## Example

sample charge config's

$\circ\ q$ $\Rightarrow$ monopole is leading term

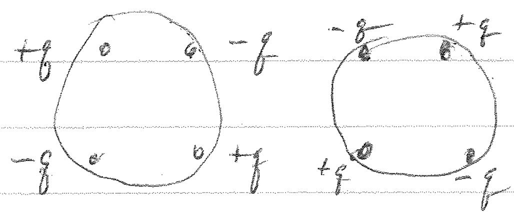
$\overset{\circ}{+q}\quad \overset{\circ}{-q}$ $\Rightarrow$ monopole $=0$ $\Rightarrow$ dipole is leading term

$\vec{p}$ is indep of origin




$\Rightarrow$ monopole $=0$ $\Rightarrow$ total dipole is sum of dipoles of individual neutral pairs


$=0$ leading term is quadrapole




when monopole $=0$ and dipole $=0$, quadrapole is indep of origin.

$\rightarrow$ total quadrapole is sum of quadrapoles of individual clusters with $q=0$ and $\vec{p}=0$

$$Q = Q_1 + Q_2$$

with $Q_2 = -Q_1$

$\Rightarrow Q = 0$ leading term is octopole

## Magnetostatics

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad , \quad \begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} = \dfrac{4\pi}{c} \vec{j} \end{cases}$$ Ampere's law (statics only!)

$$\Rightarrow \quad \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \frac{4\pi}{c} \vec{j}$$

can write $\quad \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

where by $\nabla^2 \vec{A}$ we mean $(\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z}$

$\nabla^2 \vec{A}$ only has a simple expression in Cartesian coords

Aside

If tried to write it in spherical coords, for example, one has

$$\nabla^2 \vec{A} = \nabla^2 (A_r \hat{r} + A_\theta \hat{\theta} + A_\varphi \hat{\varphi})$$

$$\cancel{= (\nabla^2 A_r)\hat{r} + A_r(\nabla^2 \hat{r}) + (\nabla^2 A_\theta)\hat{\theta} + A_\theta(\nabla^2 \hat{\theta}) + (\nabla^2 A_\varphi)\hat{\varphi} + A_\varphi(\nabla^2 \hat{\varphi})}$$

one must not forget to take the derivates of $\hat{r}, \hat{\theta}, \hat{\varphi}$ since they vary with position!

for example, $\hat{r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$

one could compute $\nabla^2 \hat{r}$ by applying $\nabla^2$ in spherical coords to each piece and summing up. Get a mess!

If work in Coulomb gauge, with $\vec{\nabla} \cdot \vec{A} = 0$, then

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \boxed{-\nabla^2 \vec{A} = \frac{4\pi}{c} \vec{j}}$$ Poisson's equation!

Many of the same methods used to solve for electrostatic $\phi$ can therefore be applied to solve for magnetostatic $\vec{A}$. But vector nature of eqn makes for complications!

For simple geometries, one can do the Coulomb-like integral

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} \qquad \text{three equations for } A_x, A_y, A_z \,!$$

for localized current sources $j(r) \to 0$ as $r \to \infty$

<u>Multipole expansion</u> - magnetic dipole moment

For a general treatment, analogous to how we did multipole expansion for electrostatics, one can use <u>vector spherical harmonics</u> - see Jackson Chpt 9.

Here we do a more straight forward approach, but only up to magnetic dipole term.

For $r \gg r'$ approx

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{(r^2 - 2\vec{r}\cdot\vec{r}' + r'^2)^{1/2}} = \frac{1}{r}\,\frac{1}{\left[1 - \frac{2\vec{r}\cdot\vec{r}'}{r^2} + \left(\frac{r'}{r}\right)^2\right]^{1/2}}$$

do Taylor series to 1st order in $\left(\frac{r'}{r}\right)$ to get

$$\frac{1}{|\vec{r}-\vec{r}'|} \simeq \frac{1}{r}\left\{1 + \frac{\vec{r}\cdot\vec{r}'}{r^2} + \cdots\right\} = \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \cdots$$

$$\vec{A}(\vec{r}) = \overset{①}{\int \frac{d^3r'}{c} \frac{\vec{j}(\vec{r}')}{r}} + \overset{②}{\int \frac{d^3r'}{c} \vec{j}(\vec{r}') \frac{(\vec{r}\cdot\vec{r}')}{r^3}} + \cdots$$

Consider term ①

$$\int d^3r\, \vec{j}(\vec{r})$$

$$\int d^3r (\vec{j}\cdot\vec{\nabla})\vec{r} \qquad \frac{\partial r_i}{\partial r_j} = \delta_{ij}$$

$$\parallel$$

write $\int d^3r\, j_i(r) = \sum_{j=1}^{3} \int d^3r\, j_j \frac{\partial r_i}{\partial r_j}$ integrate by parts

for $i^{th}$ component

$$= \sum_j \left\{ \oint_S da\, j_j\, r_i - \int d^3r \frac{\partial j_j}{\partial r_j} r_i \right\}$$

vanishes as $S \to \infty$ if $\vec{j}$ sufficiently localized ie $\vec{j}(\vec{r}) \to 0$ sufficiently fast as $r \to \infty$

vanishes in magnetostatics where $\vec{\nabla}\cdot\vec{j} = 0$

So $\int d^3r\, \vec{j}(\vec{r}) = 0$ in magnetostatics

monopole term vanishes

## term ②

$$\int d^3r \; \vec{j}(\vec{r})\,\vec{r} \qquad \text{tensor}$$

$$\frac{\partial r_i}{\partial r_k} = \delta_{ik}$$

Consider $\int d^3r \; j_i r_j = \sum_k \int d^3r \; j_k r_j \frac{\partial r_i}{\partial r_k}$ integrate by parts

$ij^{th}$ element of tensor

$$= \sum_k \left\{ \oint_S da \; j_k r_j r_i - \int d^3r \frac{\partial}{\partial r_k}(j_k r_j)\, r_i \right\}$$

vanishes as $S \to \infty$ if $\vec{j}$ sufficiently localized

$$= -\sum_k \int d^3r \left( \frac{\partial j_k}{\partial r_k} r_j r_i + j_k \frac{\partial r_j}{\partial r_k} r_i \right)$$

vanishes as $\vec{\nabla}\cdot\vec{j} = 0$ in magnetostatics $\qquad = \delta_{jk}$

$$= -\int d^3r \; j_j r_i$$

So $\int d^3r \; j_i r_j = -\int d^3r \; j_j r_i$

$$= \frac{1}{2}\int d^3r \left( j_i r_j - j_j r_i \right)$$

Going back to term ② in expansion for $\vec{A}$

So

$$\int d^3r' \; j_i(\vec{r}')(\vec{r}\cdot\vec{r}') = \sum_{j=1}^{3} r_j \int d^3r' \, j_i(\vec{r}')\, r_j'$$

$$= \sum_j \frac{1}{2}\int d^3r' \left( j_i r_j r_j' - r_j j_j r_i' \right)$$

$$= \frac{1}{2}\int d^3r' \left( j_i (\vec{r}\cdot\vec{r}') - r_i' (\vec{r}\cdot\vec{j}) \right)$$

use triple product rule

$$\vec{r} \times (\vec{r}' \times \vec{j}) = \vec{r}'(\vec{r} \cdot \vec{j}) - \vec{j}(\vec{r} \cdot \vec{r}')$$

to rewrite as

$$\int d^3r' \, \vec{j}\,(\hat{r} \cdot \vec{r}') = -\frac{1}{2} \hat{r} \times \left[ \int d^3r' \, \vec{r}' \times \vec{j}(\vec{r}') \right]$$

<u>define</u> the magnetic dipole moment as

$$\boxed{\vec{m} \equiv \frac{1}{2c} \int d^3r' \, \vec{r}' \times \vec{j}(\vec{r}')}$$

then in the magnetic dipole approx (this is the lowest non-vanishing term)

$$\vec{A}_{dip}(\vec{r}) = -\frac{\vec{r} \times \vec{m}}{r^3} = \frac{\vec{m} \times \vec{r}}{r^3} = \frac{\vec{m} \times \hat{r}}{r^2}$$

what is the magnetic field in this approx?

$$\vec{B}_{dip} = \vec{\nabla} \times \vec{A}_{dip} = \vec{\nabla} \times \left( \vec{m} \times \frac{\vec{r}}{r^3} \right)$$

to do the double cross product, it is convenient to use the Levi-Civita symbol $\varepsilon_{ijk}$ defined as

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{any two of the indices are equal} \\ +1 & ijk \text{ are an } \underline{\text{even}} \text{ permutation of } 123 \\ -1 & ijk \text{ are an } \underline{\text{odd}} \text{ permutation of } 123 \end{cases}$$

$i,j,k = 1,2,3$

In terms of Levi-Civita symbol $(\vec{A} \times \vec{B})_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} A_j B_k$

check by writing out $\vec{A} \times \vec{B}$ in terms of components

Summation conversion: when ever we have a pair of indices repeated, we sum over them, so

$$\varepsilon_{ijk} A_j B_k = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} A_j B_k$$

index $j$ appears twice
index $k$ appears twice

A <u>very</u> useful identity:

Kronecker deltas

$$\boxed{\varepsilon_{kij}\, \varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\,\delta_{jl}}$$

So we now put this to use!

$$\vec{B}_{dip} = \vec{\nabla}\times\left(\vec{m}\times \frac{\vec{r}}{r^3}\right)$$

$i$ component

$$(B_{dip})_i = \varepsilon_{ijk}\,\partial_j\, \varepsilon_{klm}\, m_l \frac{r_m}{r^3} \qquad \text{where } \partial_j \equiv \frac{\partial}{\partial r_j}$$

$$= \varepsilon_{kij}\,\varepsilon_{klm}\,\partial_j\left(m_l \frac{r_m}{r^3}\right)$$

$\varepsilon_{ijk} = \varepsilon_{kij}$ as an even permutation takes one to the other

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\,\delta_{jl}\right)\partial_j\left(m_l \frac{r_m}{r^3}\right)$$

$$= m_i\,\partial_j\left(\frac{r_j}{r^3}\right) - m_j\,\partial_j\left(\frac{r_i}{r^3}\right)$$

Now $\partial_j\left(\frac{r_j}{r^3}\right) = \vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta(\vec{r})$

$$\partial_j\left(\frac{r_i}{r^3}\right) = \frac{1}{r^3}\frac{\partial r_i}{\partial r_j} - \frac{3r_i}{r^4}\frac{\partial r}{\partial r_j}$$

by product rule and $\frac{\partial r_i}{\partial r_j} = \delta_{ij}$

Now $\frac{\partial r}{\partial x} = \frac{\partial \sqrt{x^2+y^2+z^2}}{\partial x} = \frac{x}{r}$ so $\frac{\partial r}{\partial r_j} = \frac{r_j}{r}$

don't care about this term since we only want $\vec{B}$ far away from current where $\delta(\vec{r}) = 0$.

so

$$(B_{dip})_i = \underbrace{m_i \, 4\pi \delta(\vec{r})} - m_j \left[ \frac{\delta_{ij}}{r^3} - \frac{3 r_i r_j}{r^5} \right]$$

$$= -\frac{m_i}{r^3} + \frac{3(\vec{m}\cdot\vec{r})\, r_i}{r^5}$$

$$= -\frac{m_i}{r^3} + \frac{3(\vec{m}\cdot\hat{r})\, \hat{r}_i}{r^3}$$

so

$$\boxed{\vec{B}_{dip} = \frac{3(\vec{m}\cdot\hat{r})\,\hat{r} - \vec{m}}{r^3}}$$