Quadrupole term more generally

$$\phi_{quad} = \frac{\hat{r} \cdot \overleftrightarrow{Q} \cdot \hat{r}}{2r^3}$$

since $\overleftrightarrow{Q}$ is a symmetric tensor, there is always some coordinate system in which it is diagonal. Lets work in that coordinate system. Then

$$\overleftrightarrow{Q} = \begin{pmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{pmatrix}$$

let us take $\hat{z}$ axis as the axis with the largest Q_i so $Q_3 \geq Q_1, Q_2$

in this coord system $\hat{r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$$\hat{r} \cdot \overleftrightarrow{Q} \cdot \hat{r} = Q_1 \sin^2\theta\cos^2\varphi + Q_2 \sin^2\theta \sin^2\varphi + Q_3 \cos^2\theta$$

this gives the angular variation of ϕ_{quad} as the direction of the observer varies

let us consider now averaging this over all directions

$$\frac{1}{4\pi} \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi \left[Q_1 \sin^2\theta\cos^2\varphi + Q_2 \sin^2\theta \sin^2\varphi + Q_3 \cos^2\theta \right]$$

$$= \frac{1}{4\pi} \int_0^{\pi} d\theta \sin\theta \left[\pi Q_1 \sin^2\theta + \pi Q_2 \sin^2\theta + 2\pi Q_3 \cos^2\theta \right]$$

$$= \frac{1}{4} \int_0^{\pi} d\theta \sin\theta \left[(Q_1 + Q_2) \sin^2\theta + 2Q_3 \cos^2\theta \right]$$

$$= \frac{1}{4}(Q_1 + Q_2) \int_0^{\pi} d\theta \sin^3\theta + \frac{1}{2} Q_3 \int_0^{\pi} d\theta \sin\theta \cos^2\theta$$

Now $\int_0^\pi d\theta \sin\theta \cos^2\theta = -\frac{\cos^3\theta}{3}\Big|_0^\pi = \frac{2}{3}$

$$\int_0^\pi d\theta \sin^3\theta = \int_0^\pi d\theta \sin\theta(1-\cos^2\theta)$$

$$= \int_0^\pi d\theta \sin\theta - \frac{2}{3} = -\cos\theta\Big|_0^\pi - \frac{2}{3}$$

$$= 2 - \frac{2}{3} = \frac{4}{3}$$

So angular average of $\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}$

$$= \frac{1}{4}(Q_1+Q_2)\frac{4}{3} + \frac{1}{2}Q_3\frac{2}{3} = \frac{1}{3}(Q_1+Q_2+Q_3)$$

$$= \frac{1}{3}\,\text{trace}[\overleftrightarrow{Q}]$$

But we know $\text{trace}[\overleftrightarrow{Q}] = 0$

$\Rightarrow$ angular average of Φ_{quad} always vanishes for any charge distribution!

$$\phi(\vec r) = \frac{q}{r} + \frac{\vec p \cdot \hat r}{r^2} + \frac{\hat r \cdot \overleftrightarrow{Q} \cdot \hat r}{2r^3} + \cdots$$

Note, in each term the dependence on $r = |\vec r|$ is only in the denominator. The numerators depend on the orientation of $\vec r$ via $\hat r = \frac{\vec r}{|\vec r|}$. So we can write

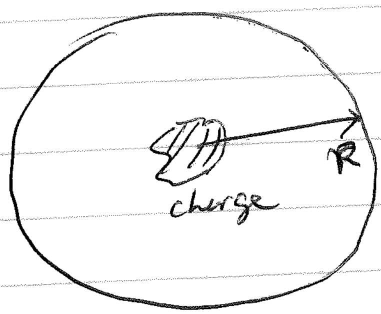
$$\phi(\vec r) = \frac{q}{r} + \sum_{n=2}^{\infty} \frac{f_n(\theta,\varphi)}{r^n}$$

where $f_n(\theta,\varphi)$ gives the dependence on the orientation of $\vec r$

Now we show that the angular average of $f_n(\theta,\varphi)$ must vanish, ie $\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi\, f_n(\theta,\varphi) = 0$

Consider the corresponding electric field $\vec E = -\vec\nabla\phi$

$$\vec E = \frac{q\hat r}{r^2} - \sum_{n=2}^{\infty} \vec\nabla\left(\frac{f_n(\theta,\varphi)}{r^n}\right)$$



consider a sphere of radius R centered on the charge distribution. Let S be the surface of this sphere. We know that

$$\oint_S da\, \hat n \cdot \vec E = 4\pi Q_{enclosed} = 4\pi q$$ ← monopole moment

since $\hat n = \hat r$

Since $$\oint_S da\, \hat n \cdot q\frac{\hat r}{r^2} = \oint_S da \frac{q}{r^2}$$

$$= 4\pi R^2 \frac{q}{R^2} = 4\pi q$$

So monopole term gives all the flux of $\vec{E}$ through the surface, and the flux of the higher terms must give zero. Since this ~~must~~ must be true for any radius R, it can only be true if each term individually gives zero flux, ie.

$$-\oint_S da\ \hat{r}\cdot\vec{\nabla}\left(\frac{f_n(\theta,\varphi)}{r^n}\right) = 0$$

but $\hat{r}\cdot\vec{\nabla} = \frac{\partial}{\partial r}$ radial directional derivative

so above is

$$-\oint_S da\ \frac{\partial}{\partial r}\left(\frac{f_n(\theta,\varphi)}{r^n}\right) = n\oint_S da\ \frac{f_n(\theta,\varphi)}{r^{n+1}}$$

$$= \frac{n}{R^{n+1}}\oint_S da\ f_n(\theta,\varphi) \qquad \text{since } r=R \text{ on } S$$

$$= \frac{n}{R^{n+1}}\, R^2\int_0^{\pi} d\theta\sin\theta\int_0^{2\pi} d\varphi\ f_n(\theta,\varphi) = 0$$

$$\text{so} \quad \int_0^{\pi} d\theta\sin\theta\int_0^{2\pi} d\varphi\ f_n(\theta,\varphi) = 0$$

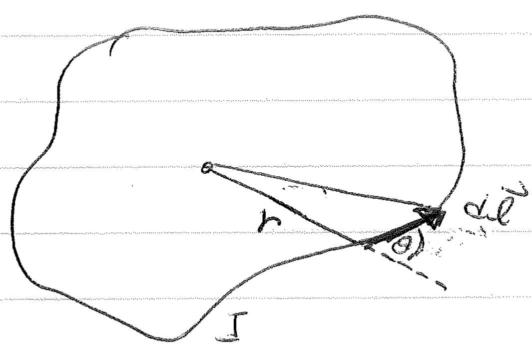
So the n-th moment contribution to Φ

$$\Phi^{(n)} = \frac{f_n(\theta,\varphi)}{r^n}$$ vanishes if we take an angular average.

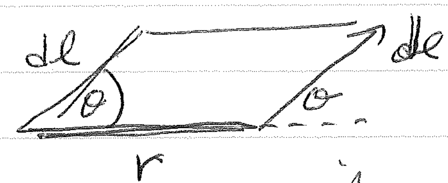
$$\boxed{\vec{B} = \frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3}}$$ same form as $\vec{E}$ from electric dipole $\vec{p}$

For a current loop in a plane (any shape loop provided it is flat)

$$\vec{m} = \frac{1}{2c}\int d^3r\ \vec{r}\times\vec{j} = \frac{1}{2c} I \oint \vec{r}\times d\vec{l}$$



area of triangle is $\frac{1}{2} r\, dl \sin\theta$
$= \frac{1}{2} |\vec{r}\times d\vec{l}|$



area of ~~rectangle~~ trapazoid is $r\, dl \sin\theta$

$$\Rightarrow \quad \vec{m} = \frac{1}{c} I (\text{area})\, \hat{M}$$

area of loop

outward normal (direction given by right hand rule with respect to direction of current) magnetic dipole moment $\vec{m}$ is independent of location of origin.

$$\vec{r}\,' = \vec{r} + \vec{d} \qquad \text{new coord}$$

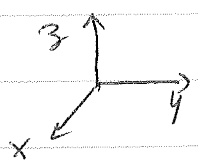
$$\vec{m}\,' = \frac{1}{2c}\int d^3r'\,(\vec{r}\,'\times\vec{J}) = \frac{1}{2c}\int d^3r\,(\vec{r}+\vec{d})\times\vec{J}$$

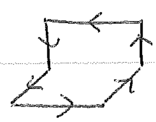
$$= \frac{1}{2c}\int d^3r\,\vec{r}\times\vec{J} + \frac{1}{2c}\vec{d}\times\left[\int d^3r\,\vec{J}\right]$$

$$\vec{m}\,' = \vec{m} + 0 \quad \text{as } \int d^3r\,\vec{J} = 0$$

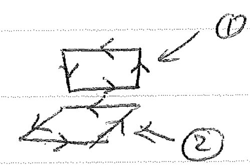
for planar loop $\vec{m} = \frac{I}{c}a\,\hat{n}$ where a = area, $\hat{n}$ = outward normal

Can also apply to get $\vec{m}$ for <u>piecewise planar loops</u>





= superposition of



$$\vec{m} = \vec{m}_1 + \vec{m}_2 \qquad \vec{m}_1 = \frac{I}{c}a_1\,\hat{x}$$

$$\vec{m}_2 = \frac{I}{c}a_2\,\hat{z}$$

$$\Rightarrow \quad \vec{m} = \frac{I}{c}(a_1\hat{x} + a_2\hat{z})$$

Boundary value problems in magnetostatics

Scalar Magnetic Potential

Because of the vector character of the equation

$$-\nabla^2 \vec{A} = \frac{4\pi}{c} \vec{J}$$

and the fact that $\nabla^2 \vec{A}$ only has a convenient representation in Cartesian coordinates, many of the methods we used to solve the scalar $-\nabla^2 \phi = 4\pi\rho$ dont work so well for magnetostatics.

However, in situations where the current $\vec{J}$ is confined to certain surfaces, we can make things much closer to the electrostatic case by using the trick of the scalar magnetic potential ϕ_M.

In regions where $\vec{J} = 0$, ie not on the certain surfaces, we have $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{B} = 0$. Since $\vec{\nabla} \times \vec{B} = 0$ in these regions we can define a scalar potential ϕ_M such that

$$\vec{B} = -\vec{\nabla} \phi_M$$

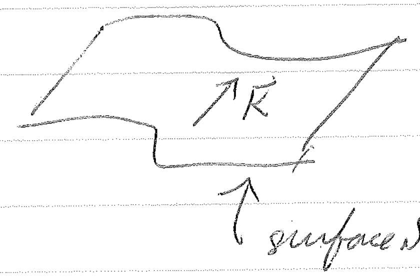
and then

$$\vec{\nabla} \cdot \vec{B} = -\nabla^2 \phi_M = 0$$

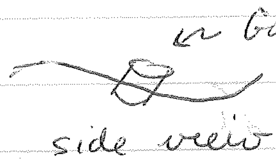
We can solve for ϕ_M as in electrostatics, and match solutions by applying appropriate boundary conditions on the current carrying surfaces.

Boundary Conditions at Sheet current

in magnetostatics $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$



surface current $\vec{K}(\vec{r})$ at pt $\vec{r}$ on surface S

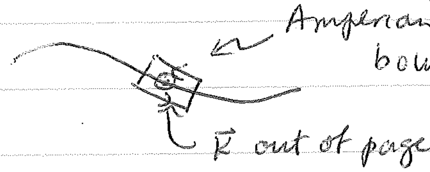


$\int_V d^3r\, \vec{\nabla}\cdot\vec{B} = 0$

top + bottom area of pill box is da
width of pill box $\to 0$

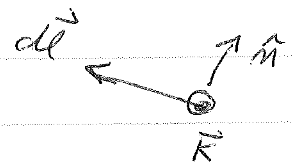
$$\Rightarrow \int_V d^3r\, \vec{\nabla}\cdot\vec{B} = \oint_S da\, \hat{n}\cdot\vec{B} = da(\vec{B}_{above} - \vec{B}_{below})\cdot\hat{n} = 0$$

<u>normal component of $\vec{B}$ is continuous</u> $(\vec{B}_{above} - \vec{B}_{below})\cdot\hat{n} = 0$



$$\int_S da\, \hat{n}\cdot(\vec{\nabla}\times\vec{B}) = \oint_C \vec{B}\cdot d\vec{l} = \frac{4\pi}{c} I_{enclosed}$$

let width of loop $\to 0$, top + bottom sides $d\vec{l}$



$$(\vec{B}_{above} - \vec{B}_{below})\cdot d\vec{l} = \frac{4\pi}{c}(\hat{n}\times d\vec{l})\cdot\vec{K}$$
$$= \frac{4\pi}{c}(\vec{K}\times\hat{n})\cdot d\vec{l}$$

$\hat{n}$ is outward normal

<u>tangential component of $\vec{B}$ has discontinuous jump $\frac{4\pi}{c}\vec{K}\times\hat{n}$</u>

Combine both results into

$$\boxed{\vec{B}_{above} - \vec{B}_{below} = \frac{4\pi}{c} \vec{K} \times \hat{n}}$$

magnetic analog of $\vec{E}_{above} - \vec{E}_{below} = 4\pi\sigma \hat{n}$

In terms of magnetic ~~vector~~ scalar potential ϕ_M

$$-\vec{\nabla}\phi_{M\,above} + \vec{\nabla}\phi_{M\,below} = \frac{4\pi}{c} \vec{K} \times \hat{n}$$

Note: ϕ_M is a calculational tool only
it does not have any direct physical
significance as does the electrostatic ϕ.
Electrostatic ϕ is related to work done
moving a charge $W_{12} = q[\phi(r_2) - \phi(r_1)]$
nothing similar for ϕ_M.
(in fact ~~magnetostatic~~ magnetic forces do no work!

$$\vec{F} = q\, \vec{v} \times \vec{B}$$

$$\Rightarrow \vec{F} \cdot \vec{v} = \frac{dW}{dt} = 0 \;)$$

Note:

ϕ_M is not necessarily continuous at surface current
Cannot do similar to electrostatics and use

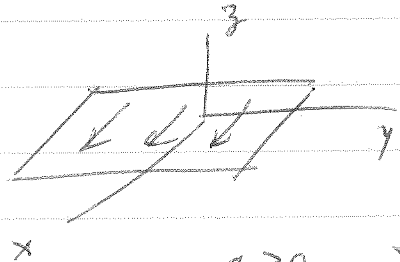
$$\phi_M(r_{above}) - \phi_M(r_{below}) = -\int_{r_{below}}^{r_{above}} \vec{B} \cdot d\vec{l}$$

Since ϕ_M is not defined on the current sheet
itself, separating "above" from "below".

example

Flat infinite plane at $z=0$ with surface current

$$\vec{K} = K\hat{x}$$



$$z \geq 0, \quad \nabla^2 \Phi_M^> = 0 \Rightarrow \Phi_M^> = a^> - b_x^> x - b_y^> y - b_z^> z$$
$$z < 0, \quad \nabla^2 \Phi_M^< = 0 \Rightarrow \Phi_M^< = a^< - b_x^< x - b_y^< y - b_z^< z$$

$$z \geq 0, \quad \vec{B}^> = -\vec{\nabla}\Phi_M^> = b_x^> \hat{x} + b_y^> \hat{y} + b_z^> \hat{z}$$
$$z < 0, \quad \vec{B}^< = -\vec{\nabla}\Phi_M^< = b_x^< \hat{x} + b_y^< \hat{y} + b_z^< \hat{z}$$

at $z=0$ $\quad \vec{B}^> - \vec{B}^< = (b_x^> - b_x^<)\hat{x} + (b_y^> - b_y^<)\hat{y} + (b_z^> - b_z^<)\hat{z}$

$$= \frac{4\pi}{c}\vec{K}\times\hat{n} = \frac{4\pi}{c}K(\hat{x}\times\hat{z}) = -\frac{4\pi}{c}K\hat{y}$$

$$\Rightarrow \quad b_x^> = b_x^< \equiv b_{x0}, \quad b_z^> = b_z^< \equiv b_{z0}, \quad b_y^> - b_y^< = -\frac{4\pi K}{c}$$

define $\left.\begin{array}{l} b_y^> = b_{y0} + \delta b_y \\ b_y^< = b_{y0} - \delta b_y \end{array}\right\} \quad \delta b_y = -\frac{2\pi}{c}K$

$$\Rightarrow \quad \vec{B}^> = \vec{B}_0 - \frac{2\pi}{c}K\hat{y} \qquad \vec{B}_0 = b_{x0}\hat{x} + b_{y0}\hat{y} + b_{z0}\hat{z}$$
$$\vec{B}^< = \vec{B}_0 + \frac{2\pi}{c}K\hat{y}$$

if $\vec{K}$ is the only source of magnetic field then $\vec{B}_0 = 0$

$$\vec{B} = \begin{cases} -\frac{2\pi}{c}K\hat{y} & z \geq 0 \\ \frac{2\pi}{c}K\hat{y} & z < 0 \end{cases}$$