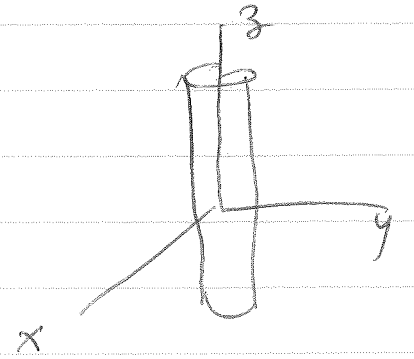
example current carrying infinite cylinder radius R



(i) $\vec{K} = K\hat{z}$ wire with surface current
(ii) $\vec{K} = K\hat{\varphi}$ solenoid

(i) $\vec{K} = K\hat{z}$ $\quad 2\pi R K = I$ total current

"guess" + show it is correct

$r>R \quad \boxed{\Phi_M = -\frac{4\pi R K}{c}\varphi}$ magnetic scalar potential $\nabla^2\Phi_M = 0$

$r<R \quad \Phi_M = 0$

$r>R \quad \vec{B} = -\vec{\nabla}\Phi_M = -\frac{1}{r}\frac{\partial \Phi_M}{\partial\varphi}\hat{\varphi} = \frac{4\pi R K}{c r}\hat{\varphi} = \boxed{\frac{2I}{cr}\hat{\varphi}}$ familiar result from Ampere

$r<R \quad \vec{B} = 0$

$\vec{B}_{above} - \vec{B}_{below} = \frac{2I}{cR}\hat{\varphi} = \frac{4\pi K}{c}\frac{R}{R}\hat{\varphi} = \frac{4\pi}{c}\vec{K}\times\hat{n}$ where $\hat{n} = \hat{r}$

as $\hat{z}\times\hat{r} = \hat{\varphi}$

Note: $\Phi_M = -\frac{4\pi}{c}RK\varphi$ is not single valued!
would not have found this using expansion
of separation of coords in polar coords
Φ_M does not need to be single valued since it has
no physical significance. Only $\vec{B} = -\vec{\nabla}\Phi_M$ is physical

(ii) $\vec{K} = K\hat{\varphi}$

$r>R \quad \Phi_M = -B_1 z$

$r<R \quad \Phi_M = -B_2 z$

$\nabla^2\Phi_M = 0$

$r>R \quad \vec{B} = -\vec{\nabla}\Phi_M = B_1\hat{z}$

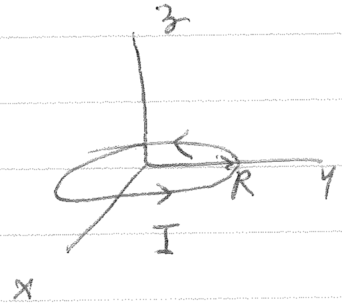
$r<R \quad \vec{B} = -\vec{\nabla}\Phi_M = B_2\hat{z}$

$$\vec{B}_{above} - \vec{B}_{below} = (B_1 - B_2)\,\hat{z} = \frac{4\pi}{c}\vec{K} \times \hat{n}$$
$$= \frac{4\pi}{c}K\,(\hat{\varphi} \times \hat{r})$$
$$= -\frac{4\pi}{c}K\,\hat{z}$$

If current in solenoid is only source of $\vec{B}$ Then expect $B_1 = 0$

$$\rightarrow \boxed{\vec{B}_2 = \frac{4\pi}{c}K\,\hat{z}} \quad \text{familiar result}$$

example circular current loop in xy plane
radius R



for $r > R$, $\vec\nabla \times \vec B = 0 \Rightarrow \vec B = -\vec\nabla \phi_M$
where $\nabla^2 \phi_M = 0$.

Try Legendre polynomial expansion for ϕ_M

$$\phi_M = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$ (A_ℓ terms vanish as want $B \to 0$ as $r \to \infty$)

$$\vec B = -\vec\nabla \phi_M = -\frac{\partial \phi_M}{\partial r}\hat r - \frac{1}{r}\frac{\partial \phi_M}{\partial \theta}\hat\theta$$

$$= \sum_\ell \left[\frac{(\ell+1)B_\ell}{r^{\ell+2}} P_\ell(\cos\theta)\hat r - \frac{B_\ell}{r^{\ell+2}} \frac{\partial P_\ell(\cos\theta)}{\partial\theta}\hat\theta \right]$$

write $\frac{\partial P_\ell}{\partial\theta} = \frac{\partial P_\ell}{\partial x}\frac{\partial x}{\partial\theta} = -\frac{\partial P_\ell}{\partial x}\sin\theta$ $\quad x = \cos\theta$
$\equiv -P_\ell' \sin\theta$

$$\vec B = \sum_\ell \left[\frac{(\ell+1)B_\ell}{r^{\ell+2}} P_\ell(\cos\theta)\hat r + \frac{B_\ell}{r^{\ell+2}}\sin\theta\, P_\ell'(\cos\theta)\,\hat\theta \right]$$

To determine the B_ℓ we compare with exact solution along $\hat z$ axis

$$\vec B(z\hat z) = \sum_\ell \frac{(\ell+1)B_\ell}{r^{\ell+2}}\hat r = \sum_\ell \frac{(\ell+1)B_\ell}{z^{\ell+2}}\hat z$$

Since $P_\ell(1) = 1$, $\sin(0) = 0$ and $P_\ell'(1)$ finite, $\hat r = \hat z$ when $\theta = 0$

exact solution on $\hat{z}$ axis:

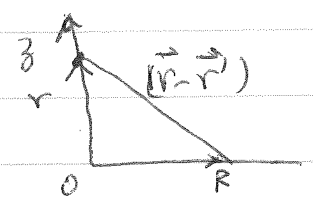
$$\vec{A} = \int \frac{d^3r'}{c} \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} \Rightarrow \vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A} = \int \frac{d^3r'}{c}\, \vec{\nabla}\times \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\vec{B} = -\int \frac{d^3r'}{c}\, \vec{j}(\vec{r}')\times\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$$\vec{B} = \int \frac{d^3r'}{c}\, \vec{j}(\vec{r}')\times\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$ Biot-Savart Law for magnetostatics

For our loop $\vec{B}(\vec{r}) = \frac{I}{c}\oint d\vec{l}'\times\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$

polar radial vector

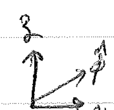


$$\vec{B}(z) = \int_0^{2\pi} d\phi \frac{R}{c}\, I\hat{\phi}\times\frac{[z\hat{z} - R\hat{r}]}{(z^2+R^2)^{3/2}}$$

$d\vec{l}' = d\phi R\hat{\phi}$

$$= \int_0^{2\pi} \frac{d\phi}{c} \frac{R(IR)\hat{z}}{(z^2+R^2)^{3/2}}$$

$\hat{\phi}\times\hat{z}$ term integrates to zero



$\hat{\phi}\times\hat{z} = \hat{r}$

$\hat{\phi}\times\hat{r} = -\hat{z}$

$$\boxed{\vec{B}(z) = \frac{2\pi R^2 I\,\hat{z}}{c(z^2+R^2)^{3/2}}}$$

to match Legendre polynomial expansion, do Taylor series expansion of above

$$\vec{B}(z) = \frac{2\pi R^2 I\,\hat{z}}{c z^3}\frac{1}{\left(1+\left(\frac{R}{z}\right)^2\right)^{3/2}} = \frac{2\pi R^2 I\,\hat{z}}{z^3}\left\{1 - \frac{3}{2}\left(\frac{R}{z}\right)^2 + \dots\right\}$$

$$= \frac{2\pi R^2 I}{c}\hat{z}\left\{\frac{1}{z^3} - \frac{3}{2}\frac{R^2}{z^5} + \dots\right\}$$

$$= \left\{\frac{B_0}{z^2} + \frac{2B_1}{z^3} + \frac{3B_2}{z^4} + \frac{4B_3}{z^5} + \dots\right\}\hat{z}$$

$$\Rightarrow B_0 = 0, \quad B_1 = \frac{\pi R^2 I}{c}, \quad B_2 = 0, \quad B_3 = -\frac{3}{4c}\pi R^2 I R^2$$

So to order $l = 3$

$$\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c}\left\{ \frac{2P_1(\cos\theta)\hat{r} + \sin\theta P_1'(\cos\theta)\hat{\theta}}{r^3} - \left[\frac{3R^2 P_3(\cos\theta)\hat{r} + \frac{3}{4}R^2 \sin\theta P_3'(\cos\theta)\hat{\theta}}{r^5}\right] + \cdots \right\}$$

$$P_1(x) = x \Rightarrow P_1'(x) = 1$$

$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x) \Rightarrow P_3'(x) = \tfrac{1}{2}(15x^2 - 3)$$

$$\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c}\left\{ \frac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3} - \left[\frac{\frac{3}{2}R^2(5\cos^3\theta - 3\cos\theta)\hat{r} + \frac{3}{8}R^2\sin\theta(15\cos^2\theta - 3)\hat{\theta}}{r^5}\right] + \cdots \right\}$$

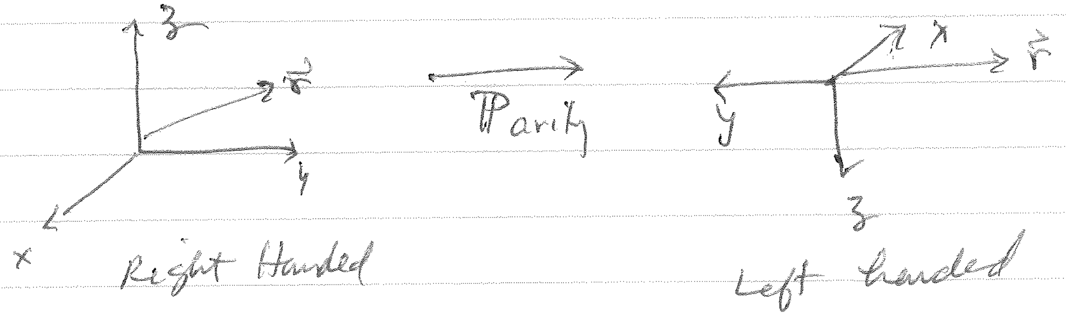
$\frac{\pi R^2 I}{c} = m$ is the magnetic dipole moment of the loop

We see that the 1st term is just the magnetic dipole approx. The 2nd term is the magnetic octopole term. Could easily get higher order terms by this method.

Compare our result above to Jackson (5-40)

Symmetry under parity transformation

vector vs. pseudovector



$\vec{r} = (x, y, z) \longrightarrow (-x, -y, -z)$

$\mathbb{P}(\vec{r}) = -\vec{r}$ position $\vec{r}$ is <u>odd</u> under parity

Any vector-like quantity that is odd under $\mathbb{P}$ is a <u>vector</u>.

<u>examples of vectors</u>

position $\vec{r}$

velocity $\vec{v} = \frac{d\vec{r}}{dt}$ since $\vec{r}$ is vector and t is scalar $\mathbb{P}(t) = t$

acceleration $\vec{a} = \frac{d\vec{v}}{dt}$

Force $\vec{F} = m\vec{a}$ since $\vec{a}$ is vector and m is scalar

momentum $\vec{p} = m\vec{v}$ since $\vec{v}$ is vector and m is scalar

electric field $\vec{F} = q\vec{E}$ since $\vec{F}$ is vector and q is scalar $\mathbb{P}(q) = q$

current $\vec{j} = \sum_i q_i \vec{v}_i \, \delta(\vec{r} - \vec{r}_i(t))$

any vector-like quantity that is <u>even</u> under $\mathbb{P}$ is a <u>psuedovector</u>

angular momentum $\vec{L} = \vec{r} \times \vec{p}$ since $\vec{r} \to -\vec{r}$ and $\vec{p} \to -\vec{p}$,

$\vec{L} \to \vec{L}$ under $\mathbb{P}$

$\vec{L}$ is even under $\mathbb{P}$

magnetic field $\vec{F} = q\vec{v} \times \vec{B}$ since $\vec{F}$ and $\vec{v}$ are vectors and q is scalar, $\vec{B}$ must be ~~psuedov~~ psuedovector.

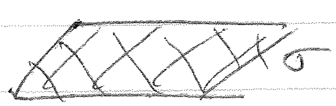
cross product of any two vectors is a psuedovector

" " " vector and psuedovector is a vector

when solving for $\vec{E}$, it can only ~~be~~ ~~E~~ made up of <u>vectors</u> that exist in the problem

When solving for $\vec{B}$, it can only be made up of <u>psuedovectors</u> that exist in the problem

<u>ex</u> charged plane

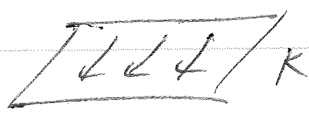


only directions in problem is normal $\hat{n}$

$\hat{n}$ is a <u>vector</u>

$\vec{E} \propto \hat{n}$

surface current



only directions are the <u>vectors</u> $\hat{n}$ and $\vec{K}$. But $\vec{B}$ can only be made of <u>psuedovectors</u>

$\Rightarrow \vec{B} \propto (\vec{K} \times \hat{n})$

Dielectrics + Magnetic Materials - Macroscopic Maxwell Equ

~~Dielectrics~~

Maxwell's equations apply exactly to the true microscopic electric and magnetic fields that arise from all charges and currents

$$\vec{\nabla}\cdot\vec{b} = 0 \qquad \vec{\nabla}\times\vec{e} + \frac{1}{c}\frac{\partial\vec{b}}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{e} = 4\pi\rho_0 \qquad \vec{\nabla}\times\vec{b} = \frac{4\pi}{c}\vec{j}_0 + \frac{1}{c}\frac{\partial\vec{e}}{\partial t}$$

where $\vec{e}$ and $\vec{b}$ are microscopic fields from total charge density ρ_0 and current density $\vec{j}_0$.

However, in most problems involving macroscopic objects, if we took ρ_0 and $\vec{j}_0$ to describe charge + current of each individual atom in a material, then they, and the resulting $\vec{e}$ and $\vec{b}$ would be enormously complicated functions varying rapidly over distances $\sim 10^{-8}$ cm and times $\sim 10^{-16}$ sec.

In classical E+M we are generally concerned with phenomena that vary extremely slowly compared to these length + time scales,