example current carrying infinite cylinder radiator

\[ \mathbf{\hat{r}} = K \mathbf{\hat{z}} \]

wire with surface current

\[ \mathbf{R} = K \mathbf{\hat{\phi}} \]

solenoid

\( r > R \)

\[ \phi_M = -4\pi r K \Phi \]

\[ \phi_M = 0 \]

magnetic scalar potential \( \nabla^2 \phi_M = 0 \)

\( r < R \)

\[ \mathbf{B} = -\nabla \phi_M = -\frac{1}{r} \frac{\partial \phi_M}{\partial \rho} \mathbf{\hat{\rho}} = \frac{4\pi r K}{2} \frac{\mathbf{\hat{\phi}}}{r} = \frac{2I}{cR} \frac{\mathbf{\hat{\phi}}}{c} \]

familiar result from Ampere

\( \mathbf{B} \) below \( \mathbf{B} \) above

\[ \frac{2I}{cR} \mathbf{\hat{\phi}} = \frac{4\pi r K}{2} \mathbf{\hat{\phi}} = \frac{4\pi r K}{2} \mathbf{\hat{r}} \]

where \( \mathbf{\hat{r}} = \mathbf{\hat{\phi}} \)

as \( \mathbf{\hat{r}} \times \mathbf{\hat{\phi}} = \mathbf{\hat{\phi}} \)

Note: \( \phi_M = -4\pi r K \Phi \) is not single valued. It would not have found this using expansion of separation of coords in polar coords

\( \phi_M \) does not need to be single valued since it has no physical significance, only \( \mathbf{B} = -\nabla \phi_M \) is physical

\( r > R \)

\[ \phi_M = -B_1 \mathbf{\hat{\rho}} \]

\[ \nabla^2 \phi_M = 0 \]

\( r < R \)

\[ \phi_M = -B_2 \mathbf{\hat{r}} \]

\( r > R \)

\[ \mathbf{B} = -\nabla \phi_M = B_1 \mathbf{\hat{\rho}} \]

\( r < R \)

\[ \mathbf{B} = -\nabla \phi_M = B_2 \mathbf{\hat{r}} \]
Above \( \mathbf{B}_{\text{above}} \)  
\( = (\mathbf{B}_1 - \mathbf{B}_2) \frac{2}{3} \)
\( = \frac{4\pi K}{2} \mathbf{\hat{r}} \times \mathbf{\hat{m}} \)
\( = \frac{4\pi K}{2} (\mathbf{\hat{r}} \times \mathbf{\hat{m}}) \)
\( = -\frac{4\pi K}{2} \mathbf{\hat{z}} \)

If current in solenoid is only source \( \mathbf{B} \). Then expect \( \mathbf{B}_1 \equiv 0 \)

\( \Rightarrow \boxed{\mathbf{B}_2 = \frac{4\pi K}{2} \mathbf{\hat{z}}} \)

Familiar result
Example: circular current loop in xy plane

\[ 3 \text{ rad} \ldots R \]

For \( r > R \), \( \vec{\nabla} \times \vec{B} = 0 \) \( \implies \vec{B} = -\vec{\nabla} \phi_M \)

Where \( \nabla^2 \phi_M = 0 \).

Try Legendre polynomial expansion for \( \phi_M \)

\[ \phi_M = \sum_{l=0}^{\infty} \frac{B_{2l}}{r^{l+1}} P_{2l}(\cos \theta) \quad (A_0 \text{ terms vanish as want } \vec{B} \to 0 \text{ as } r \to \infty) \]

\[ \vec{B} = -\vec{\nabla} \phi_M = -\frac{\partial \phi_M}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial \phi_M}{\partial \theta} \hat{\theta} \]

\[ = \sum_{l=0}^{\infty} \left[ \frac{(l+1)B_{2l}}{r^{l+1}} P_{2l}(\cos \theta) \hat{r} - \frac{B_{2l}}{r^{l+2}} \frac{2l}{\partial \theta} P_{2l}(\cos \theta) \hat{\theta} \right] \]

Write

\[ \frac{\partial P_{2l}}{\partial \theta} = \frac{\partial P_0}{\partial x} \cdot \frac{\partial x}{\partial \theta} = -\frac{\partial P_0}{\partial x} \sin \theta \quad x = \cos \theta \]

\[ = -P_{2l} \sin \theta \]

\[ \vec{B} = \sum_{l=0}^{\infty} \left[ \frac{(l+1)B_{2l}}{r^{l+2}} P_{2l}(\cos \theta) \hat{r} + \frac{B_{2l}}{r^{l+2}} \sin \theta P_{2l}(\cos \theta) \hat{\theta} \right] \]

To determine the \( \vec{B}_k \) we compare with exact solution along \( \hat{\theta} \) axis

\[ \vec{B}(\vec{r}, \hat{\theta}) = \sum_{l=0}^{\infty} \frac{(l+1)B_{2l}}{r^{l+2}} P_{2l}(\cos \theta) \hat{r} = \sum_{l=0}^{\infty} \frac{(l+1)B_{2l}}{r^{l+2}} \]

Since \( P_0(1) = 1 \), \( \sin(0) = 0 \) and \( P_0'(1) \) finite, \( \hat{r} = \hat{\theta} \) when \( \theta = 0 \)
exact solution on $\hat{z}$ axis:

$$\vec{A} = \int d^3r' \frac{\vec{f}(r')}{|r-r'|} \Rightarrow \vec{B}(r) = \nabla \times \vec{A} = \int d^3r' \frac{\nabla \times \vec{f}(r')}{|r-r'|}$$

$$\vec{B} = -\int d^3r' \frac{\vec{f}(r') \times \nabla}{|r-r'|^3}$$ Biot–Savart law for magnetostatics.

For each loop $\vec{B}(r) = \frac{1}{c} \int \int d\mathbf{r} \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$ polar vector vector

$$\vec{B}(z) = \int_0^{2\pi} d\phi \int_0^\infty dR \frac{\hat{z} \times \vec{R}}{c (z^2 + R^2)^{3/2}}$$

$$= \frac{2\pi}{c} \int_0^\infty dR \frac{R (IR) \hat{z}}{(z^2 + R^2)^{3/2}}$$

$$\vec{B}(z) = \frac{2\pi R^2 I \hat{z}}{c (z^2 + R^2)^{3/2}}$$

To match Legendre polynomial expansion, do Taylor series expansion

If above

$$\vec{B}(z) = \frac{2\pi R^2 I \hat{z}}{c (z^2 + R^2)^{3/2}}$$

$$= \frac{2\pi R^2 I \hat{z}}{c} \left\{ \frac{1}{3} - \frac{3}{2} \frac{R^2}{3^2} + \ldots \right\}$$

$$= \frac{2\pi R^2 I \hat{z}}{c} \left\{ \frac{1}{3} - \frac{3}{2} \frac{R^2}{3^2} + \ldots \right\}$$

$$= \frac{2 R^2 I \hat{z}}{3} \left\{ \frac{1}{3} - \frac{3}{2} \frac{R^2}{3^2} + \ldots \right\}$$

$$= \frac{2 R^2 I \hat{z}}{3} \left\{ \frac{1}{3} - \frac{3}{2} \frac{R^2}{3^2} + \ldots \right\}$$

$$= \frac{B_0}{3^2} + 2B_1 + \frac{3 B_2}{3^3} + \frac{4 B_3}{3^4} + \ldots$$
\[ B_0 = 0, \quad B_1 = \frac{\pi R^2 I}{c}, \quad B_2 = 0, \quad B_3 = -\frac{3\pi R^2 I R^2}{4c} \]

So to order \( I = 3 \)

\[
\vec{B}(r) = \frac{\pi R^2 I}{c} \left\{ \frac{2P_1(\cos \theta) \hat{r} + \sin \theta P_1'(\cos \theta) \hat{\theta}}{r^3} \right\} - \left[ \frac{3R^2 P_3(\cos \theta) \hat{r} + \frac{3}{2} R^2 \sin \theta P_3'(\cos \theta) \hat{\theta}}{r^5} \right] + \ldots \]

\[ P_1(x) = x \quad \Rightarrow \quad P_1'(x) = 1 \]

\[ P_3(x) = \frac{1}{2}(5x^3 - 3x) \quad \Rightarrow \quad P_3'(x) = \frac{1}{2}(15x^2 - 3) \]

\[
\vec{B}(r) = \frac{\pi R^2 I}{c} \left\{ \frac{2 \cos \theta \hat{r} + \sin \theta \hat{\theta}}{r^3} \right\} - \left[ \frac{\frac{3}{2} R^2 (5 \cos^3 \theta - 3 \cos \theta) \hat{r} + \frac{3}{8} R^2 \sin \theta (15 \cos^2 \theta - 3) \hat{\theta}}{r^5} \right] + \ldots \]

\[ \frac{\pi R^2 I}{c} = m \] is the magnetic dipole moment of the loop

We see that the 1st term is just the magnetic dipole approx. The 2nd term is the magnetic octopole term. Could easily get higher order terms by this method.

Compare our result above to Jackson (5-40)
Symmetry under parity transformation

vector vs. pseudo vector

\[ \vec{r} = (x, y, z) \rightarrow (-x, y, -z) \]

\[ P(\vec{r}) = -\vec{r} \] position \( \vec{r} \) is odd under \( P \)

Any vector-like quantity that is odd under \( P \) is a **vector**.

**Examples of vectors**

position \( \vec{r} \)

velocity \( \vec{v} = \frac{d\vec{r}}{dt} \) since \( \vec{v} \) is vector and \( t \) is scalar

acceleration \( \vec{a} = \frac{d\vec{v}}{dt} \)

Force \( \vec{F} = m\vec{a} \) since \( \vec{a} \) is vector and \( m \) is scalar

momentum \( \vec{p} = m\vec{v} \) since \( \vec{v} \) is vector and \( m \) is scalar

Electric field \( \vec{E} = g \vec{E} \) since \( \vec{E} \) is vector and \( g \) is scalar

current \( \vec{j} = \sum_i \vec{j}_i \delta \) \( \delta (\vec{r} - \vec{r}_i(t)) \)
any vector-like quantity that is even under \( P \) is a pseudo-vector.

Angular momentum \( \vec{L} = \vec{r} \times \vec{p} \) since \( \vec{r} \rightarrow -\vec{r} \) and \( \vec{p} \rightarrow \vec{p} \),

\( \vec{L} \rightarrow \vec{L} \) under \( P \)

\( \vec{L} \) is even under \( P \)

Magnetic field \( \vec{F} = q \vec{v} \times \vec{B} \) since \( \vec{F} \) and \( \vec{v} \) are vectors and \( q \) is scalar, \( \vec{B} \) must be pseudo-vector.

cross product of any two vectors is a pseudo-vector.

" vector and pseudo-vector is a vector.

When solving for \( \vec{E} \), it can only be made up of vectors that exist in the problem.

When solving for \( \vec{B} \), it can only be made up of pseudo-vectors that exist in the problem.

\( E \) charged plane

\( k \times k \) only directions in problem is normal \( \hat{m} \)

\( \hat{m} \) is a vector

\( \vec{E} \times \hat{m} \)

Surface current

only directions are the vectors \( \hat{m} \) and \( \hat{k} \). But \( \vec{B} \) can only be made of pseudo-vectors

\( \Rightarrow \vec{B} \propto (\hat{k} \times \hat{m}) \)
Dielectrics and Magnetic Materials - Macroscopic
Maxwell Equation

Maxwell's equations apply exactly to the true microscopic electric and magnetic fields that arise from all charges and currents.

\[ \nabla \cdot \mathbf{b} = 0 \quad \nabla \times \mathbf{e} + \frac{1}{c} \frac{\partial \mathbf{b}}{\partial t} = 0 \]

\[ \nabla \cdot \mathbf{e} = 4\pi \rho_0 \quad \nabla \times \mathbf{b} = \frac{4\pi}{c} \mathbf{J}_0 + \frac{i \omega}{c} \mathbf{e} \]

where \( \mathbf{e} \) and \( \mathbf{b} \) are microscopic fields from total charge density \( \rho_0 \) and current density \( \mathbf{J}_0 \).

However, in most problems involving macroscopic objects, if we took \( \rho_0 \) and \( \mathbf{J}_0 \) to describe charge and current of each individual atom in a material, then they, and the resulting \( \mathbf{e} \) and \( \mathbf{b} \) would be enormously complicated functions varying rapidly over distances \( \sim 10^{-8} \) cm and times \( \sim 10^{-10} \) sec.

In classical E&M, we are generally concerned with phenomena that vary extremely slowly compared to these length and time scales.