

when atoms have intrinsic magnetic moments due to electron spin, we can add these to \vec{M} in obvious way

when molecules are neutral, $q_n=0$, the "bound current" is given by

$$\vec{j}_{\text{bound}} = \sum_n \langle \vec{j}_n \rangle = c \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t}$$

Note that the $\frac{\partial \vec{P}}{\partial t}$ term is crucial to give conservation of bound charge

$$\begin{aligned} \vec{\nabla} \cdot \vec{j}_{\text{bound}} &= c \vec{\nabla} \cdot (\vec{\nabla} \times \vec{M}) + \vec{\nabla} \cdot \frac{\partial \vec{P}}{\partial t} \\ &= 0 + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{P}) \end{aligned}$$

$$= -\frac{\partial \rho_{\text{bound}}}{\partial t} \quad \text{where } \rho_{\text{bound}} = -\vec{\nabla} \cdot \vec{P} \text{ is bound charge density}$$

$$\text{So } \boxed{\vec{\nabla} \cdot \vec{j}_{\text{bound}} + \frac{\partial \rho_{\text{bound}}}{\partial t} = 0}$$

and bound charge is conserved.

Since total average charge must be conserved, i.e.

$$\vec{\nabla} \cdot \langle \vec{j}_0 \rangle - \frac{\partial \langle \rho_0 \rangle}{\partial t} = 0, \quad \text{and } \langle \vec{j}_0 \rangle = \vec{j} + \vec{j}_{\text{bound}}$$

↑
free current

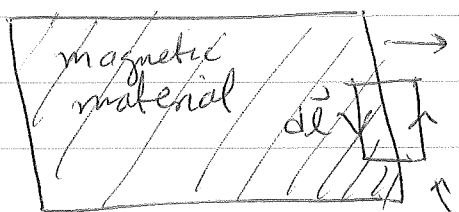
$$\langle \rho_0 \rangle = \rho + \rho_{\text{bound}}$$

↑
free charge

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0}$$

free charge is also conserved

At a surface of a magnetic material



\hat{m} outward normal to surface

take $\hat{z} \equiv d\vec{l} \times \hat{m}$ out of page

Amperian loop C bounding surface S of area da

$$\begin{aligned}
 c \int_S da \hat{z} \cdot (\nabla \times \vec{M}) &= \int_S da \hat{z} \cdot \left[\vec{j}_{\text{bound}} - \frac{\partial \vec{P}}{\partial t} \right] = da \hat{z} \cdot \left[\vec{j}_{\text{bound}} - \frac{\partial \vec{P}}{\partial t} \right] \\
 &= (d\vec{l} \times \hat{m}) \cdot \vec{K}_{\text{bound}} \quad \text{as width of loop} \\
 &= (\hat{m} \times \vec{K}_{\text{bound}}) \cdot d\vec{l} \quad \begin{matrix} \rightarrow 0 \\ \frac{\partial \vec{P}}{\partial t} \text{ term} \\ \text{vanishes as } da \rightarrow 0 \end{matrix}
 \end{aligned}$$

But by Stokes theorem

$$c \int_S da \hat{z} \cdot (\nabla \times \vec{M}) = c \oint_C d\vec{l} \cdot \vec{M} = c d\vec{l} \cdot \vec{M} \quad \begin{matrix} \text{since width} \rightarrow 0 \\ \text{and } \vec{M} = 0 \text{ outside} \end{matrix}$$

$$\Rightarrow c d\vec{l} \cdot \vec{M} = (\hat{m} \times \vec{K}_{\text{bound}}) \cdot d\vec{l} \quad \text{for any } d\vec{l} \text{ in plane of surface}$$

$$\Rightarrow c \vec{M}_t = \hat{m} \times \vec{K}_{\text{bound}}$$

where \vec{M}_t is component of \vec{M} tangential to the surface (since \vec{K}_b is in plane of surface, $\hat{m} \times \vec{K}_b$ is also entirely in the plane of the surface)

$$\Rightarrow c \hat{m} \times \vec{M}_t = c \hat{m} \times \vec{M} = \hat{m} \times (\hat{m} \times \vec{K}_{\text{bound}}) = -\vec{K}_{\text{bound}}$$

$$\Rightarrow \begin{cases} \vec{K}_{\text{bound}} = c \vec{M} \times \hat{m} \\ \vec{j}_{\text{bound}} = c \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \end{cases}$$

Total bound charge vanishes (for neutral molecules)

$$Q_{\text{bound}} = \int_V d^3r \rho_{\text{bound}} + \int_S da \sigma_{\text{bound}}$$

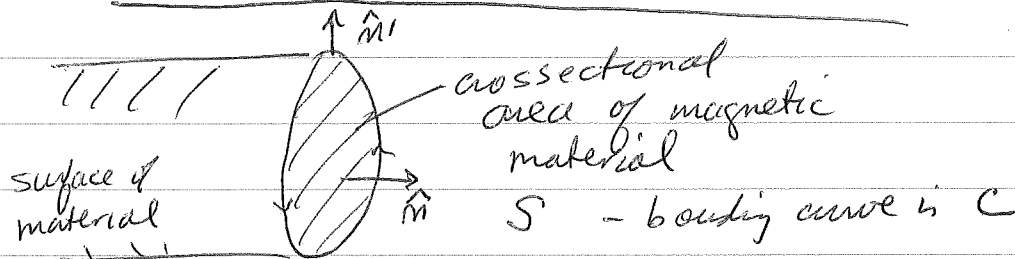
\uparrow vol of dielectric \leftarrow surface of dielectric

$$= \int_V d^3r (-\vec{\nabla} \cdot \vec{P}) + \int_S da \hat{n} \cdot \vec{P}$$

but by Gauss theorem $\int_V d^3r \vec{\nabla} \cdot \vec{P} = \int_S da \hat{n} \cdot \vec{P}$

$$\text{so } Q_{\text{bound}} = - \int_S da \hat{n} \cdot \vec{P} + \int_S da \hat{n} \cdot \vec{P} = 0$$

Total bound current vanishes



\hat{n} is normal to cross-section
 \hat{n}' is normal to surface

total current flowing through S is

$$\int_S da \hat{n} \cdot \vec{j}_{\text{bound}} + \int_C dl \vec{K}_{\text{bound}} \cdot \hat{n}$$

$$= c \int_S da \hat{n} \cdot (\vec{\nabla} \times \vec{M}) + c \int_C dl \hat{n} \cdot (\vec{M} \times \hat{n}')$$

$$= c \int_0 dl \vec{e} \cdot \vec{M} + c \int_C dl (\hat{n}' \times \hat{n}) \cdot \vec{M}$$

$= -\hat{x}$ unit tangent, $d\vec{l} = dl \hat{x}$

$$= c \int_C d\vec{l} \cdot \vec{M} - c \int_C d\vec{l} \cdot \vec{M} = 0$$

Macroscopic Maxwell Equations

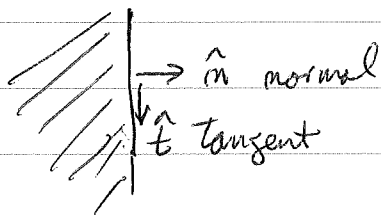
$$\begin{aligned}\vec{\nabla} \cdot \vec{B} &= 0 & \vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} &= 0 \\ \vec{\nabla} \times \vec{H} &= \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t} & \vec{\nabla} \cdot \vec{D} &= 4\pi \rho\end{aligned}$$

where ρ and \vec{j} are macroscopic charge + current densities
do not include bound charges or currents

$$\begin{aligned}\vec{D} &= \vec{E} + 4\pi \vec{P} & \vec{P} & \text{is polarization density} \\ \vec{H} &= \vec{B} - 4\pi \vec{M} & \vec{M} & \text{is magnetization density}\end{aligned}$$

Boundary conditions for statics

electrostatics: at surface of a dielectric, or at interface between two different dielectrics



$$\vec{\nabla} \times \vec{E} = 0 \Rightarrow \hat{t} \cdot \vec{E}_{\text{above}} = \hat{t} \cdot \vec{E}_{\text{below}}$$

tangential component \vec{E} is continuous

proof same as before

$$\vec{\nabla} \cdot \vec{D} = 4\pi \rho \Rightarrow \hat{n} \cdot (\vec{D}_{\text{above}} - \vec{D}_{\text{below}}) = 4\pi \sigma$$

normal component of \vec{D} jumps by $4\pi \sigma$

magneto statics: at surface or interface of magnetic materials

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \hat{n} \cdot \vec{B}_{\text{above}} - \hat{n} \cdot \vec{B}_{\text{below}} = 0$$

normal component of \vec{B} is continuous

proof same as before

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{j} \Rightarrow \hat{t} \cdot (\vec{H}_{\text{above}} - \vec{H}_{\text{below}}) = \frac{4\pi}{c} (\vec{k} \times \hat{n}) \cdot \hat{t}$$

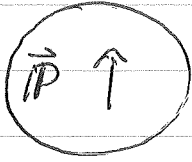
tangential component of \vec{H} jumps by $\frac{4\pi}{c} \vec{k} \times \hat{n}$

if $\sigma = 0$, i.e. no free surface charge, then $\hat{n} \cdot \vec{D}$ continuous

if $\vec{k} = 0$, i.e. no free surface current, then $\hat{t} \cdot \vec{H}$ continuous

Examples

① Uniformly polarized sphere of radius R $\vec{P} = P\hat{z}$



bound charge $\rho_b = -\vec{\nabla} \cdot \vec{P} = 0$ as \vec{P} constant

$$\sigma_b = \hat{n} \cdot \vec{P} = \hat{r} \cdot \vec{P} = P \cos \theta$$

we saw earlier that a sphere with surface charge $\sigma(\theta) = \sigma_0 \cos \theta$ gives an electric field like a pure dipole for $r > R$, and is constant for $r < R$.

$$\vec{E}(\vec{r}) = \begin{cases} \left(\frac{4}{3}\pi R^3 P\right) \left[\frac{2\cos\theta \hat{r} + \sin\theta \hat{\theta}}{r^3} \right] & r > R \\ -\frac{4\pi P}{3} \hat{z} & r < R \end{cases}$$

total dipole moment is $\vec{p} = \frac{4}{3}\pi R^3 P \hat{z}$

check behavior at boundary

Tangential component \vec{E}

$$\vec{E}_{\text{above}}^{\parallel} = \left(\frac{4}{3}\pi R^3 P\right) \frac{\sin\theta}{R^3} \hat{\theta} = \frac{4\pi P}{3} \sin\theta \hat{\theta}$$

$$\vec{E}_{\text{below}}^{\parallel} = -\frac{4\pi P}{3} (\hat{z} \cdot \hat{\theta}) \hat{\theta} = \frac{4\pi P}{3} \sin\theta \hat{\theta}$$

\Rightarrow Tangential component \vec{E} is continuous

normal component of \vec{D}

outside: $\vec{P} = 0 \Rightarrow \vec{D} = \vec{E}$

$$\Rightarrow \hat{n} \cdot \vec{D} = \hat{r} \cdot \vec{E} = \left(\frac{4}{3}\pi R^3 P\right) \frac{2\cos\theta \hat{r}}{R^3} = \frac{8}{3}\pi P \cos\theta$$

$$\text{inside: } \vec{E} = -\frac{4\pi\vec{P}}{3} \Rightarrow \vec{P} = -\frac{3}{4\pi}\vec{E}$$

$$\vec{D} = \vec{E} + 4\pi\vec{P} = \vec{E} - 3\vec{E} = -2\vec{E} = \frac{8\pi P}{3}\hat{z}$$

$$\hat{m} \cdot \vec{D} = \hat{r} \cdot \left(\frac{8\pi P}{3} \hat{z} \right) = \frac{8\pi P}{3} \cos\theta$$

\Rightarrow normal component \vec{D} is continuous

Note: normal component of \vec{E} should jump by $4\pi\sigma_b = 4\pi P \cos\theta$

to check this: $\hat{m} \cdot \vec{E} = \hat{r} \cdot \left(-\frac{4}{3}\pi P \hat{z} \right) = -\frac{4}{3}\pi P \cos\theta$

$$\hat{m} \cdot (\vec{E}^{\text{above}} - \vec{E}^{\text{below}}) = \frac{8}{3}\pi P \cos\theta + \frac{4}{3}\pi P \cos\theta$$

$$= \frac{12}{3}\pi P \cos\theta = 4\pi P \cos\theta = \sigma_b(\theta)$$

(2) Uniformly magnetized sphere of radius R $\vec{M} = M \hat{z}$



bound current $\vec{j}_b = c \vec{\nabla} \times \vec{M} = 0$ as \vec{M} constant
 $\vec{K}_b = c \vec{M} \times \hat{m} = cM (\hat{z} \times \hat{r})$
 $= cM \sin \theta \hat{\phi}$

We saw earlier that a sphere with surface current $\vec{K}_b = K_0 \sin \theta \hat{\phi}$ gives a magnetic field that is pure dipole for $r > R$, and is constant for $r < R$.

$$\vec{B}(\vec{r}) = \begin{cases} \left(\frac{4}{3} \pi R^3 M \right) \left[\frac{2 \cos \theta \hat{r} + \sin \theta \hat{\theta}}{r^3} \right] & r > R \\ \frac{8}{3} \pi M \hat{z} & r < R \end{cases}$$

total dipole moment is $\vec{m} = \frac{4}{3} \pi R^3 \vec{M}$

check behavior at boundary

normal component of \vec{B}

$$\hat{m} \cdot \vec{B}^{\text{above}} = \hat{r} \cdot \vec{B}^{\text{above}} = \frac{8}{3} \pi M \cos \theta$$

$$\hat{m} \cdot \vec{B}^{\text{below}} = \hat{r} \cdot \vec{B}^{\text{below}} = \frac{8}{3} \pi M (\hat{r} \cdot \hat{z}) = \frac{8}{3} \pi M \cos \theta$$

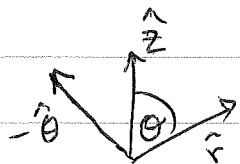
\Rightarrow normal component of \vec{B} is continuous

tangential component of \vec{H}

outside: $\vec{M} = 0 \Rightarrow \vec{H} = \vec{B}$

$$\vec{H}_{\text{above}}^t = \left(\frac{4}{3} \pi M\right) \sin \theta \hat{\theta}$$

inside: $\vec{H} = \vec{B} - 4\pi \vec{M} = \vec{B} - 4\pi \left(\frac{3}{8\pi} \vec{B}\right) = \vec{B} - \frac{3}{2} \vec{B} = -\frac{1}{2} \vec{B}$
 $= -\frac{4\pi M}{3} \hat{z}$



so $\vec{H}_{\text{below}}^t = -\frac{4\pi}{3} M (\hat{z} \cdot \hat{\theta}) = \frac{4\pi}{3} M \sin \theta \hat{\theta}$

\Rightarrow tangential component \vec{H} is continuous

Note: tangential component \vec{B} should jump by $\frac{4\pi}{c} \vec{K}_b \times \hat{m} = 4\pi M \sin \theta \hat{\theta}$

inside:

to check: $\vec{B}_{\text{below}}^t = \frac{8}{3} \pi M (\hat{z} \cdot \hat{\theta}) \hat{\theta} = -\frac{8}{3} \pi M \sin \theta \hat{\theta}$

$$\vec{H}_{\text{above}}^t = \vec{B}_{\text{above}}^t \Rightarrow \vec{B}_{\text{above}}^t - \vec{B}_{\text{below}}^t = \frac{4}{3} \pi M \sin \theta \hat{\theta} + \frac{8}{3} \pi M \sin \theta \hat{\theta}$$

$$= 4\pi M \sin \theta \hat{\theta} = \frac{4\pi}{c} \vec{K}_b \times \hat{r}$$

since $\hat{\theta} = \hat{\phi} \times \hat{r}$

Linear Materials

Macroscopic Maxwell Equations

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho \quad \vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}$$

where ρ and \vec{j} are macroscopic charge & current densities
and

$$\begin{aligned} \vec{D} &= \vec{E} + 4\pi\vec{P} & \vec{P} &\text{ is polarization density} \\ \vec{H} &= \vec{B} - 4\pi\vec{M} & \vec{M} &\text{ is magnetization density} \end{aligned}$$

To close these equations, we will in general need to know how \vec{P} and \vec{M} are related to the \vec{E} and \vec{B} in the material.

In some materials, there can be a finite \vec{P} or \vec{M} even if \vec{E} and \vec{B} are zero:

Ferromagnet: \vec{M} can be non zero even if $\vec{B} = 0$

Ferroelectric: \vec{P} can be non zero even if $\vec{E} = 0$

But more common are linear materials in which, for small \vec{E} and \vec{B} , one has $\vec{P} \propto \vec{E}$ and $\vec{M} \propto \vec{B}$.

linear dielectric

$$\vec{P} = \chi_e \vec{E}$$

χ_e is "electric susceptibility"
 $\chi_e > 0$ for statics

$$\vec{D} = \vec{E} + 4\pi \vec{P} = (1 + 4\pi \chi_e) \vec{E}$$

$$\vec{D} = \epsilon \vec{E} \quad \text{with} \quad \epsilon = 1 + 4\pi \chi_e$$

ϵ is the dielectric constant

linear magnetic materials

$$\vec{M} = \chi_m \vec{H}$$

χ_m is "magnetic susceptibility"

$\chi_m > 0 \Rightarrow$ paramagnetic

$\chi_m < 0 \Rightarrow$ diamagnetic

$$\vec{H} = \vec{B} - 4\pi \vec{M} = \vec{B} - 4\pi \chi_m \vec{H}$$

$$\vec{B} = (1 + 4\pi \chi_m) \vec{H}$$

$$\vec{B} = \mu \vec{H} \quad \text{with} \quad \mu = 1 + 4\pi \chi_m$$

μ is magnetic permeability

For statics, $\chi_e > 0$ and χ_m (or alternatively ϵ and μ) are constants depending on the material.

When we consider dynamics we will see that ϵ becomes a function of frequency.

Clausius - Mossotti equation

Electric susceptibility + atomic polarizability

If a field \vec{E}_{loc} is applied to an atom, it gets polarized

$$\vec{p} = \alpha \vec{E}_{loc}$$

↑ ↑ ↑
atomic dipole "local field" - field the atom sees
moment atomic polarizability

α is what one calculates from a microscopic theory

If $\vec{E}_{loc} = \vec{E}$ the average field in the material then electric susceptibility given by

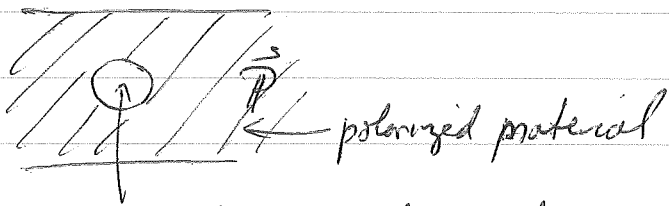
$$\vec{P} = n \vec{p} = n \alpha \vec{E}_{loc} = n \alpha \vec{E} = \chi_e \vec{E}$$

$\Rightarrow \chi_e = n \alpha$ where $n =$ density of atoms

But a more careful consideration shows $\vec{E}_{loc} \neq \vec{E}$
The average field \vec{E} includes the electric field created by the polarized atom itself. \vec{E}_{loc} , the local field the atom sees, should exclude its own self field.

$$\vec{E} = \vec{E}_{loc} + \vec{E}_{atom}$$

↑ ↑ ↑
average average average field of
field field excluding the atom
 atom



cut out sphere whose volume is V_m
the volume per atom

\vec{E}_{loc} is field excluding the field of the polarized sphere of volume V_m .

\vec{E}_{atom} is field of the polarized sphere

$$\vec{E}_{atom} = -\frac{4\pi\vec{P}}{3} = -\frac{4\pi m\vec{p}}{3}$$

$$\vec{E}_{loc} = \vec{E} - \vec{E}_{atom} = \vec{E} + \frac{4\pi\vec{P}}{3} = \vec{E} + \frac{4\pi m\vec{p}}{3}$$

$$\vec{p} = \alpha \vec{E}_{loc} = \alpha \left(\vec{E} + \frac{4\pi m\vec{p}}{3} \right) = \alpha \vec{E} + \frac{4\pi m\alpha}{3} \vec{p}$$

$$\vec{p} = \frac{\alpha}{1 - \frac{4\pi m\alpha}{3}} \vec{E}$$

$$\vec{P} = m\vec{p} = \frac{\alpha m}{1 - \frac{4\pi m\alpha}{3}} \vec{E} = \chi_e \vec{E}$$

$$\chi_e = \frac{m\alpha}{1 - \frac{4\pi m\alpha}{3}}$$

or solve for α in terms of ϵ

$$\chi_e = \frac{m\alpha}{1 - \frac{4\pi}{3}m\alpha} \Rightarrow \chi_e - \frac{4\pi}{3}m\alpha\chi_e = \alpha m$$

$$\Rightarrow \alpha = \frac{\chi_e}{m(1 + \frac{4\pi}{3}\chi_e)}$$

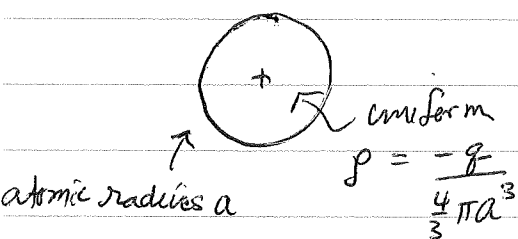
$$\epsilon = 1 + 4\pi\chi_e \Rightarrow \alpha = \frac{\epsilon - 1}{4\pi m} \frac{1}{(1 + \frac{\epsilon - 1}{3})}$$

relates atomic polarizability to measured dielectric constant

$$\alpha = \frac{3}{4\pi m} \left(\frac{\epsilon - 1}{\epsilon + 2} \right)$$

Clausius Mossotti
or Lorentz-Lorenz equation

single model for α



field inside is $E(r) = \frac{4\pi\rho}{3} r \hat{r}$

In external field E_0 , electron cloud will be displaced from cen by distance d . We can determine d by making a balance of forces: $\Rightarrow qE_0 = q \frac{4\pi\rho d}{3}$

$$\chi_e = \frac{m a^3}{1 - \frac{4\pi}{3} m a^3}$$

$$\rho = qd = \frac{3}{4\pi\rho} q E_0 = \frac{3}{4\pi\rho} \left(\frac{4\pi\rho a^3}{3} \right) q E_0$$

$$= a^3 E_0 \Rightarrow \alpha = a^3$$

if $f = m \frac{4\pi}{3} a^3$ fraction of vol that is occupied by atoms

$$\chi_e = \frac{1}{4\pi} \frac{3f}{1-f}$$