ith component of integrand on right hand side is (E part only)

\[ E_i \delta_j E_j = E_{ijk} E_j E_{km} \delta_k E_m \]

\[ = E_i \delta_j E_j - (\delta_{ij} \delta_{jm} - \delta_{im} \delta_{jk}) E_j \delta_k E_m \]

\[ = E_i \delta_j E_j - E_j \delta_i E_j + E_j \delta_j E_i \]

\[ = \delta_j (E_i E_j - \frac{1}{2} \delta_{ij} E^2) \]

Define Maxwell's stress tensor

\[ T_{ij} = \frac{1}{4\pi} \left[ E_i E_j + \nabla_i \cdot B_j - \frac{1}{2} \delta_{ij} (E^2 + B^2) \right] \]

(Note \( T_{ij} = T_{ji} \))

Symmetric tensor

Then

\[ \frac{d}{dt} \vec{p}_{\text{mech}} + \frac{d}{dt} \iiint d^3 r \vec{T}_i = \iiint d^3 r \frac{\partial}{\partial x_i} T_{ij} \]

\[ = \oint_S d\alpha \cdot T_{ij} \hat{n}_j \]

\[ \frac{d}{dt} \vec{p}_{\text{mech}} + \frac{d}{dt} \iiint d^3 r \vec{T} = \oint_S d\alpha \cdot \vec{T} \cdot \hat{n} \]

- \( T_{ij} \) gives the flow of the \( i \)th component of electromagnetic field momentum through an element of surface area \( S \) to direction \( \hat{e}_j \)

For static situations where \( \vec{F} = \frac{d}{dt} \vec{p}_{\text{mech}} = \oint_S d\alpha \cdot \vec{T} \cdot \hat{n} \)

Gives electromagnetic force on the surface.
Note: $\frac{d\mathbf{P}}{dx}$ is equal to the total electromagnetic force on the volume $V$.

Hence we can write

$$\mathbf{F}_{EM} = \oint_S d\mathbf{a} \, \mathbf{T} \cdot \hat{n} - d \int_V d\mathbf{r} \, \mathbf{T}$$

for static situations, the 2nd term vanishes and

$$\mathbf{F}_{EM} = \oint_S d\mathbf{a} \, \mathbf{T} \cdot \hat{n} \quad \mathbf{T}_{ij} \text{ is } i^{th} \text{ component of static force on unit area with normal } \hat{e}_j$$

this is origin of the term "stress" tensor, \(\mathbf{T}\) is like the stress tensor of an elastic medium, \(T_{xx}, T_{yy}, T_{zz}\) are like pressure, off diagonal elements are like shear stresses
Force on a conductor surface.

- Surface charge on conductor

\[ \vec{F} = \frac{1}{4\pi} \left[ \vec{E}_{\text{above}} \cdot \hat{m} - \vec{E}_{\text{below}} \cdot \hat{m} \right] \]

\( \vec{F} = 0 \) as \( \vec{E} = 0 \) inside conductor

For conductor surface

\[ \hat{m} \cdot \vec{E}_{\text{above}} = \frac{4\pi}{5} \] (since \( \vec{E}_{\text{below}} = 0 \))

and tangential component \( \vec{E} = 0 \)

\[ \Rightarrow \vec{E} = \frac{4\pi}{5} \hat{m} \]

So

\[ \vec{F} = \frac{1}{4\pi} \left[ (4\pi \hat{m}) (4\pi \hat{m}) - \frac{1}{2} \hat{m} (4\pi \hat{m})^2 \right] \]

\[ \vec{F} = \frac{\hat{m}}{4\pi} \left[ (4\pi \hat{m})^2 - \frac{1}{2} (4\pi \hat{m})^2 \right] = 2\pi \sigma^2 \hat{m} \]

Force per unit area:

\[ \vec{f} = 2\pi \sigma^2 \hat{m} = \frac{1}{2} \sigma \vec{E} \]

Note factor \( \frac{1}{2} \). Namely one might have thought \( \vec{F} = \sigma \vec{E} \), but need to exclude self field of charge on surface from acting on itself. See also Jackson pp 42 for another approach.
Capacitance

Consider a set of conductors with potential $\phi(\vec{r}) = V_i$ fixed on conductor $i$.

(Also need condition on $\phi(\vec{r}) \to 0$ if system is not enclosed.)

From uniqueness theorem we know that specifying the $V_i$ on each conductor is enough to determine the potential $\phi(\vec{r})$ everywhere. We can write this potential in the following form.

Let $\phi_i(\vec{r})$ be the solution to the boundary value problem

$\nabla^2 \phi_i(\vec{r}) = 0 \quad \text{and} \quad \phi_i(\vec{r}) = \begin{dcases} 
1 & \text{if } \vec{r} \text{ on surface of conductor } i \\
0 & \text{if } \vec{r} \text{ on surface of any other conductor } j, \ j \neq i
\end{dcases}$

Then by superposition

$\phi(\vec{r}) = \sum_i V_i \phi_i(\vec{r})$

is solution to the problem $\nabla^2 \phi = 0$ and $\phi(\vec{r}) = V_i$ for $\vec{r}$ on surface of conductor $i$.

The surface charge density at $\vec{r}$ on surface of conductor $i$ is

$\sigma_i(\vec{r}) = \frac{1}{4\pi} \frac{\partial \phi(\vec{r})}{\partial n} = -\frac{1}{4\pi} \sum_i V_i \frac{\partial \phi_i(\vec{r})}{\partial n}$

where $\frac{\partial \phi}{\partial n} = (\nabla \phi) \cdot \hat{n}$ is the derivative normal to the surface at point $\vec{r}$. 
The total charge on conductor (i) is

\[ Q_i = \oint_{S_i} \sigma_i(\mathbf{r}) \, d\mathbf{a} = -\frac{1}{4\pi} \sum_j V_j \int_{S_i} \frac{\partial \phi_j}{\partial n} \, d\mathbf{a} \]

Surface of conductor (i)

Define \( C_{ij} = -\frac{1}{4\pi} \int_{S_i} \frac{\partial \phi_j}{\partial n} \, d\mathbf{a} \)

the \( C_{ij} \) depend only on the geometry of the conductors.

Then we have

\[ Q_i = \sum_j C_{ij} V_j \]

\( C_{ij} \) is the capacitance matrix

The charge on conductor (i) is a linear function of the potentials \( V_j \) on the conductors (j).

Since we know that specifying the \( Q_i \) that is on each conductor will uniquely determine \( \phi(\mathbf{r}) \) and hence the potential \( V_i \) on each conductor, the capacitance matrix is invertible.

\[ V_i = \sum_j \left( C^{-1} \right)_{ij} Q_j \]

The electrostatic energy of the conductors is then

\[ \mathcal{E} = \frac{1}{2} \int \partial \mathbf{r} \phi \mathbf{r} = \frac{1}{2} \sum_i Q_i V_i = \frac{1}{2} \sum_i \sum_j C_{ij} V_i V_j = \frac{1}{2} \sum C_{ij} Q_i Q_j \]

\[ \left[ V \cdot C^{-1} Q \right] \]
Compute to define capacitance of two conductors

\[ C = \frac{Q}{V_1 - V_2} \]

when conductor (1) has charge \( Q \)

conductor (2) has charge \(-Q\)

\( V_1 - V_2 \) is potential difference

between the two conductors.

all other conductors fixed at \( V_c = 0 \)

We can determine \( C \) in terms of the elements of the

matrix \( C_{ij} \).

\[ \begin{align*}
Q_1 &= C_{11}V_1 + C_{12}V_2 \\
-Q_2 &= C_{21}V_1 + C_{22}V_2
\end{align*} \]

\[ \Rightarrow V_2 = -\left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}}\right)V_1 \]

\[ \Rightarrow Q = \left[ C_{11} - C_{12}\left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}}\right)\right]V_1 \]

\[ V_1 - V_2 = \left[ 1 + \left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}}\right)\right]V_1 \]

\[ C = \frac{Q}{V_1 - V_2} = \frac{C_{11} - C_{12}\left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}}\right)}{1 + \left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}}\right)} \]

\[ C = \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{11} + C_{12} + C_{21} + C_{22}} \]

Capacitance can also be defined when the space
between the conductors is filled with a dielectric \( \varepsilon \).

In this case, if \( Q_c \) is the free charge, then \( Q_c/\varepsilon \) is
the effective total charge to use in computing \( \phi \).
\[ \Rightarrow \oint \frac{d\ell}{\ell} = \sum_j C_{ij}^{(0)} V_j \]

where \( C_{ij}^{(0)} \) are capacitances appropriate to a vacuum between the conductors.

\[ \Rightarrow \oint_i = \sum_j \varepsilon \, C_{ij}^{(o)} V_j \]

\[ = \sum_j C_{ij} V_j \quad \text{where} \quad C_{ij} = \varepsilon C_{ij}^{(o)} \]

the capacitance is increased by a factor the dielectric constant \( \varepsilon \).
Consider a set of current carrying loops $C_i$ with currents $I_i$.

\[ \Phi_i = \sum_j M_{ij} I_j \]

where $M_{ij} = \oint_{C_i} \oint_{C_j} \frac{d\ell_i \cdot d\ell_j}{\ell_i \cdot \ell_j}$

is the mutual inductance of loops $(i)$ and $(j)$.

The magnetic flux through loop $i$ is

\[ \Phi_i = \oint_{C_i} \hat{\mathbf{n}} \cdot \mathbf{B} = \oint_{C_i} \hat{\mathbf{n}} \cdot \nabla \times \mathbf{A} = \oint_{C_i} d\mathbf{\ell} \cdot \mathbf{A} \]

\[ = \oint_{C_i} d\mathbf{\ell} \cdot \mathbf{A} \]

\[ = \int d\mathbf{\ell} \cdot \mathbf{A} \]

In Coulomb gauge, we can write the magnetic vector potential $\mathbf{A}$ from these current loops as

\[ \mathbf{A}(\mathbf{r}) = \int \frac{d^3 \mathbf{r}'}{4\pi \mathbf{r} \cdot \mathbf{r}'} = \sum_i \frac{I_i}{\epsilon_0 c} \oint_{C_i} \frac{d\mathbf{\ell}'}{\mathbf{r} \cdot \mathbf{r}'} \]

Integrate over loop $C_i$. Integrated variable is $\mathbf{r}'$. The magnetic flux through loop $i$ is

\[ \Phi_i = \sum_j M_{ij} I_j \]

where $M_{ij} = \oint_{C_i} \oint_{C_j} \frac{d\ell_i \cdot d\ell_j}{\ell_i \cdot \ell_j}$
\[ L_i = M_{ii} \text{ is self-inductance of loop } (i) \]

The sign convention in the above is that \( \Phi_i \) is computed in direction given by right hand rule, according to the direction taken for current in loop \((i)\).

\[ \Phi_i \]

Magnetic static energy

\[ E = \frac{1}{2c} \int d^3r \, \hat{j} \cdotp \hat{A} = \frac{1}{2c} \sum_i \int d^2r \, \hat{A} \cdotp \hat{I}_i \]

\[ = \frac{1}{2c} \sum_i \Phi_i \hat{I}_i \]

\[ E = \frac{1}{2} \sum_{i,j} M_{ij} \hat{I}_i \hat{I}_j \]
Electromagnetic Waves in a Vacuum

No sources \( \hat{f} = 0, \ \hat{g} = 0 \)

1) \( \vec{\nabla} \cdot \vec{E} = 0 \)
2) \( \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \)
3) \( \vec{\nabla} \cdot \vec{B} = 0 \)
4) \( \vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \)

\[ \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \vec{\nabla} \times \vec{\nabla} \vec{E} = -\frac{1}{c^2} \frac{\partial E}{\partial t} (\vec{\nabla} \times \vec{B}) \]

\[ \Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \]

\[ -\vec{\nabla}^2 \vec{E} = -\frac{1}{c^2} (\vec{\nabla} \times \vec{B}) = -\frac{1}{c^2} \left( \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right) \]

\[ \vec{\nabla}^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \]

Similarly

\[ \vec{\nabla}^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \]

\( \text{wave equation} \)

\( \text{wave speed} \approx c \)

Note: In MKS units, above wave equation looks like

\[ \vec{\nabla}^2 \vec{E} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \]

It was noticed that the speed of electromagnetic wave,

\[ \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 \text{ m/s} \]

was the same as the speed of light! This observation was a key element in showing that light was in fact electromagnetic waves.
Harmonic Plane waves

\[ \vec{E}(\vec{r},t) = \text{Re} \left\{ \vec{E}_k e^{i(k \cdot \vec{r} - \omega t)} \right\} \]
\[ \vec{B}(\vec{r},t) = \text{Re} \left\{ \vec{B}_k e^{i(k \cdot \vec{r} - \omega t)} \right\} \]

\( \vec{k} \) is wave vector
\( \omega \) is angular frequency
\( \nu = \frac{\omega}{2\pi} \) is frequency
\( T = \frac{1}{\nu} \) is period
\( \lambda = \frac{2\pi}{|k|} \) is wavelength

\[ \begin{vmatrix} \vec{E}_k \\ \vec{B}_k \end{vmatrix} \begin{vmatrix} |E_k| \\ |B_k| \end{vmatrix} \]

\[ \vec{E}(\vec{r} + \lambda \hat{k},t) = \vec{E}(\vec{r},t) \] periodic in space with period \( \lambda \)
\[ \vec{E}(\vec{r},t + \tau) = \vec{E}(\vec{r},t) \] periodic in time with period \( \tau \)

A "plane wave" \( \Rightarrow \vec{E}(\vec{r},t) \) is constant in space on planes with normal \( \hat{m} \parallel \hat{k} \).

Properties of EM plane waves

\[ \nabla \cdot \vec{E} = 0 \quad \Rightarrow \quad \text{Re} \left\{ \vec{E}_k \cdot \nabla \vec{E} \right\} e^{i(k \cdot \vec{r} - \omega t)} \]
\[ = \text{Re} \left\{ i \vec{E}_k \cdot \hat{k} e^{i(k \cdot \vec{r} - \omega t)} \right\} = 0 \]
\[ \Rightarrow \vec{E}_k \cdot \hat{k} = 0 \]

amplitude is orthogonal to \( \vec{k} \)

\[ \nabla \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{B}_k \cdot \hat{k} = 0 \]

amplitude orthogonal to \( \vec{k} \)