→ as with $\vec{j}$ and $\vec{E}$, relation between $\vec{D}$ and $\vec{E}$ is <u>non-local</u> in time

$$\vec{D}(t) \neq \varepsilon \vec{E}(t)$$

rather

$$\vec{D}(t) = \int_{-\infty}^{\infty} dt' \, \vec{E}(t') \, \tilde{\varepsilon}(t-t')$$

Fourier transf of $\varepsilon(\omega)$

Ampere's law is

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}$$

becomes $\frac{1}{\mu} \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \int_{-\infty}^{\infty} dt' \, \vec{E}(t') \frac{d}{dt} \tilde{\varepsilon}(t-t')$

integro-differential equation!

Maxwells equations only look simple when expressed in terms of Fourier transforms

$$\vec{E}(\vec{r},t) = \vec{E}_\omega \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$
$$\vec{B}(\vec{r},t) = \vec{B}_\omega \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$
$$\vec{D}(\vec{r},t) = \vec{D}_\omega \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$
$$\vec{H}(\vec{r},t) = \vec{H}_\omega \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

Maxwell's Equ for <u>source free</u> system $\rho = \vec{j} = 0$

$$\vec{\nabla}\cdot\vec{D} = 0, \quad \vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{c\,\partial t}, \quad \vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{c\,\partial t}$$

assume μ is true constant - not freq dependent

dielectric response is $\vec{D}_\omega = \varepsilon(\omega)\vec{E}_\omega$

Then for the Fourier amplitudes of the fields, Maxwell's Equations become

transverse polarized

$$\begin{aligned}
&1)\quad i\vec{k}\cdot\vec{D}_\omega = i\,\varepsilon(\omega)\,\vec{k}\cdot\vec{E}_\omega = 0 \quad \Rightarrow \quad \boxed{\vec{k}\perp\vec{E}_\omega} \quad (\text{unless } \varepsilon(\omega)=0)\\
&2)\quad i\vec{k}\cdot\vec{B}_\omega = 0 \quad \Rightarrow \quad \boxed{\vec{k}\perp\vec{B}_\omega}\\
&3)\quad i\vec{k}\times\vec{E}_\omega = i\frac{\omega}{c}\vec{B}_\omega\\
&4)\quad i\vec{k}\times\vec{H}_\omega = -i\frac{\omega}{c}\vec{D}_\omega \quad \Rightarrow \quad \frac{i\vec{k}}{\mu}\times\vec{B}_\omega = -i\frac{\omega}{c}\varepsilon(\omega)\vec{E}_\omega
\end{aligned}$$

$$\vec{k}\times(3) = i\vec{k}\times(\vec{k}\times\vec{E}_\omega) = i\frac{\omega}{c}\vec{k}\times\vec{B}_\omega$$

$$\Rightarrow -ik^2\vec{E}_\omega = -i\frac{\omega^2}{c^2}\varepsilon(\omega)\mu\,\vec{E}_\omega \qquad \text{using (4)}$$

$$\boxed{k^2 = \frac{\omega^2}{c^2}\varepsilon(\omega)\mu}$$ <u>dispersion relation</u>

<u>Note:</u> $\dfrac{\omega}{|k|} = \dfrac{c}{\sqrt{\varepsilon(\omega)\mu}}$ varies with ω.

There is not a single phase velocity.

$\Rightarrow \vec{E}$ is <u>not</u> in general a solution of a wave equation - different frequencies travel with different speeds

Since $\varepsilon(\omega)$ is complex $\quad \varepsilon(\omega) = \varepsilon_1(\omega) + i\varepsilon_2(\omega)$

$\Rightarrow$ wave vector also complex $\quad$ For $\vec{k} = k\hat{z}$

$$k = k_1 + ik_2 = \pm \frac{\omega}{c}\sqrt{\mu}\,\sqrt{\varepsilon_1 + i\varepsilon_2}$$

$$\vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \vec{E}_\omega e^{i[(k_1+ik_2)z - \omega t]}$$
$$= \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$$

k_1 determines the oscillation of the wave

k_2 determines the decay or <u>attenuation</u> of the wave as it propagates into the material

<u>phase velocity</u> $\quad v_p \equiv \frac{\omega}{k_1}$

<u>index of refraction</u> $\quad n = \frac{c}{v_p} = \frac{ck_1}{\omega}$

<u>group velocity</u> $v_g \qquad \frac{1}{v_g} = \frac{dk_1}{d\omega}$

<u>Magnetic field</u>: $\quad \vec{B}_\omega = \frac{c\vec{k}}{\omega} \times \vec{E}_\omega$

for $\vec{k} = k\hat{z}$, $\quad \vec{B}_\omega = \frac{c(k_1 + ik_2)}{\omega}\,\hat{z} \times \vec{E}_\omega$

if $\quad k_1 + ik_2 = \sqrt{k_1^2 + k_2^2}\, e^{i\delta} \qquad \delta = \arctan\left(\frac{k_2}{k_1}\right)$

$$= |k| e^{i\delta}$$

$$\vec{B}_\omega = \frac{c|k|}{\omega}\,\hat{z} \times \vec{E}_\omega\, e^{i\delta}$$

phase shift

$$\vec{B}(\vec{r},t) = \frac{c|k|}{\omega} (\hat{z}\times\vec{E}_\omega)\, e^{-k_2 z}\, e^{i(k_1 z - \omega t + \delta)}$$

Physical fields - take real parts

$$\vec{E}(\vec{r},t) = \vec{E}_\omega\, e^{-k_2 z} \cos(k_1 z - \omega t)$$

$$\vec{B}(\vec{r},t) = (\hat{z}\times\vec{E}_\omega)\frac{c|k|}{\omega}\, e^{-k_2 z} \cos(k_1 z - \omega t + \delta)$$

Conclusions

1) $\vec{E}$ and $\vec{B}$ $\perp$ $\vec{k}$ transverse polarized
2) $\vec{E} \perp \vec{B}$
3) amplitude ratio $\frac{|\vec{B}|}{|\vec{E}|} = \frac{c|k|}{\omega} = \sqrt{|\varepsilon(\omega)|\mu}$
4) $\vec{B}$ is shifted in phase with respect to $\vec{E}$ by phase shift $\delta = \arctan(k_2/k_1)$
5) waves decay as they propagate $e^{-k_2 z}$

consequence of complex $\varepsilon(\omega)$

If $\varepsilon_2 = 0$, ie $\varepsilon(\omega)$ is real, and if $\varepsilon > 0$, then $k_2 = 0 \Rightarrow$ no decay, no phase shift

consequences of frequency dependence of $\varepsilon(\omega)$

6) $\vec{E}(t)$ and $\vec{D}(t)$ non locally related in time
7) waves of different ω travel with different $v_p = \omega/k_1$
8) dispersion - wave pulses do not travel with v_p and they spread as the propagate

pulses travel with group velocity $v_g = \frac{d\omega}{dk}$
(see Quantum Mechanics discussion)

$v_g < v_p$ "normal dispersion"

$v_g > v_p$ "anomalous dispersion"

index of refraction

$$\frac{1}{v_g} = \frac{dk_1}{d\omega} = \frac{d}{d\omega}\left[\frac{\omega}{c} n\right]$$

$$\frac{1}{v_g} = \frac{n}{c} + \frac{\omega}{c}\frac{dn}{d\omega} = \frac{1}{v_p} + \frac{\omega}{c}\frac{dn}{d\omega}$$

$$v_g = \frac{v_p}{1 + \frac{v_p}{c}\,\omega \frac{dn}{d\omega}}$$

$\Rightarrow$ when $\begin{cases} \frac{dn}{d\omega} > 0, & v_g < v_p \quad \text{normal dispersion} \\ \frac{dn}{d\omega} < 0, & v_g > v_p \quad \text{anomalous dispersion} \end{cases}$

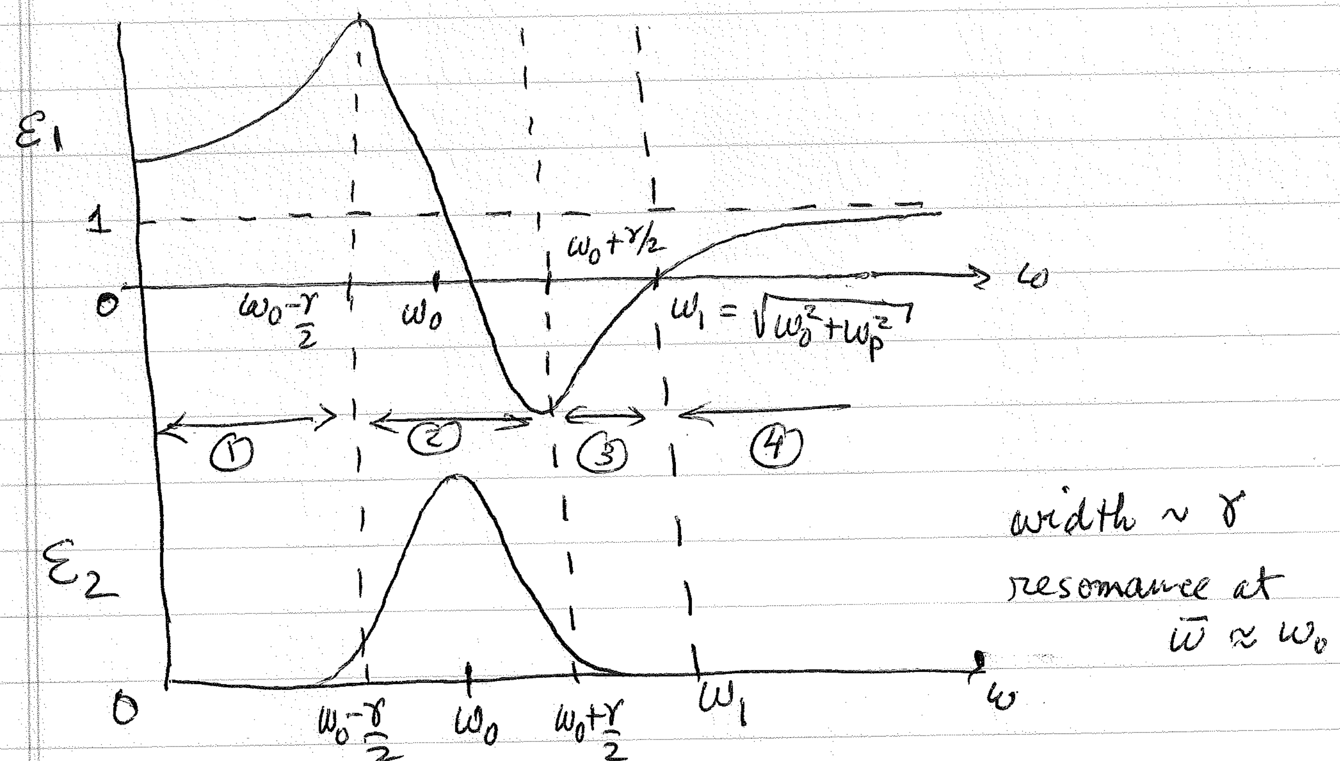
For our simple model: $\varepsilon = 1 + 4\pi\chi \simeq 1 + 4\pi M\alpha$

$$\varepsilon(\omega) = 1 + \frac{4\pi M e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

$$\varepsilon_1 = 1 + \frac{4\pi M e^2}{m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}$$

$$\varepsilon_2 = \frac{4\pi M e^2}{m} \frac{\omega\gamma}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}$$

Define $\omega_p = \sqrt{\frac{4\pi M e^2}{m}}$ the "plasma frequency"



$$k = k_1 + ik_2 = \pm \frac{\omega}{c}\sqrt{\mu}\sqrt{\varepsilon_1 + i\varepsilon_2}$$

$$k^2 = k_1^2 - k_2^2 + 2ik_1k_2 = \frac{\omega^2}{c^2}\mu(\varepsilon_1 + i\varepsilon_2)$$

equate real and imaginary pieces and solve for k_1 and k_2

$$k_1 = \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\sqrt{\varepsilon_1^2+\varepsilon_2^2} + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$k_2 = \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\sqrt{\varepsilon_1^2+\varepsilon_2^2} - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

Regions of different behavior

Regions 1 and 4 – transparent propagation

$\varepsilon_1 > 0$, $\varepsilon_1 \gg \varepsilon_2$ expand the $\sqrt{\ }$ in Taylor series

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_1\left(1+\frac{1}{2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right) + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\varepsilon_1 + \frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_1}\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu\varepsilon_1} \quad + \text{ small correction}$$

$$k_2 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_1\left(1+\frac{1}{2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right) - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_1}\right]^{1/2} = k_1\left(\frac{\varepsilon_2}{2\varepsilon_1}\right) \ll k_1$$

So $k_2 \ll k_1$ small attenuation

$\Rightarrow$ medium is transparent

Note: $v_p = \dfrac{\omega}{k_1} = \dfrac{c}{n} = \dfrac{c}{\sqrt{\varepsilon_1\mu}}$

in region 1, $\varepsilon_1 > 1 \Rightarrow v_p < c$

in region 4, $\varepsilon_1 < 1 \Rightarrow v_p > c$!

but $v_g < c$ always!

Region ② $\quad \omega \approx \omega_0 \quad$ resonant absorption

$$\varepsilon_2 \approx \frac{\omega_p^2}{\omega_0 \gamma} = \left(\frac{\omega_p}{\omega_0}\right)^2 \left(\frac{\omega_0}{\gamma}\right) \gg 1$$

for a sharp resonance with $\gamma \ll \omega_0$

$$\varepsilon_1 \simeq 1$$

So $\quad \varepsilon_2 \gg \varepsilon_1$

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_2\left(1 + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2\right) + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_2 + \frac{1}{4}\frac{\varepsilon_1^2}{\varepsilon_2} + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\sqrt{\tfrac{1}{2}\varepsilon_2}$$

$$k_2 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_2\left(1 + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2\right) - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_2 + \frac{1}{4}\frac{\varepsilon_1^2}{\varepsilon_2} - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu}\sqrt{\tfrac{1}{2}\varepsilon_2}$$

$k_1 \approx k_2 \quad$ strong attenuation

wave excits atoms at resonance $\Rightarrow$ large atomic displacements $\Rightarrow$ media absorbs most energy from the wave $\Rightarrow$ wave decays rapidly, decreases factor $\frac{1}{e^{2\pi}}$ within one wavelength of propagation.

Region 3 $\quad \varepsilon_1 < 0, \quad |\varepsilon_1| \gg \varepsilon_2$

total reflection

width of region 3 is

$$\omega_1 - \omega_0 = \sqrt{\omega_0^2 + \omega_p^2} - \omega_0 \sim \omega_p \sim \sqrt{N}$$

increases with atomic density $\qquad$ as $\omega_p \gg \omega_0$

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}|\varepsilon_1| + \frac{1}{4}\frac{\varepsilon_2^2}{|\varepsilon_1|} + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

cancel as $|\varepsilon_1| = -\varepsilon_1$

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu|\varepsilon_1|}\ \frac{\varepsilon_2}{2|\varepsilon_1|}$$

$$k_2 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}|\varepsilon_1| + \frac{1}{4}\frac{\varepsilon_2^2}{|\varepsilon_1|} - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu|\varepsilon_1|}$$

$$\frac{k_2}{k_1} = \frac{2|\varepsilon_1|}{\varepsilon_2} \gg 1$$

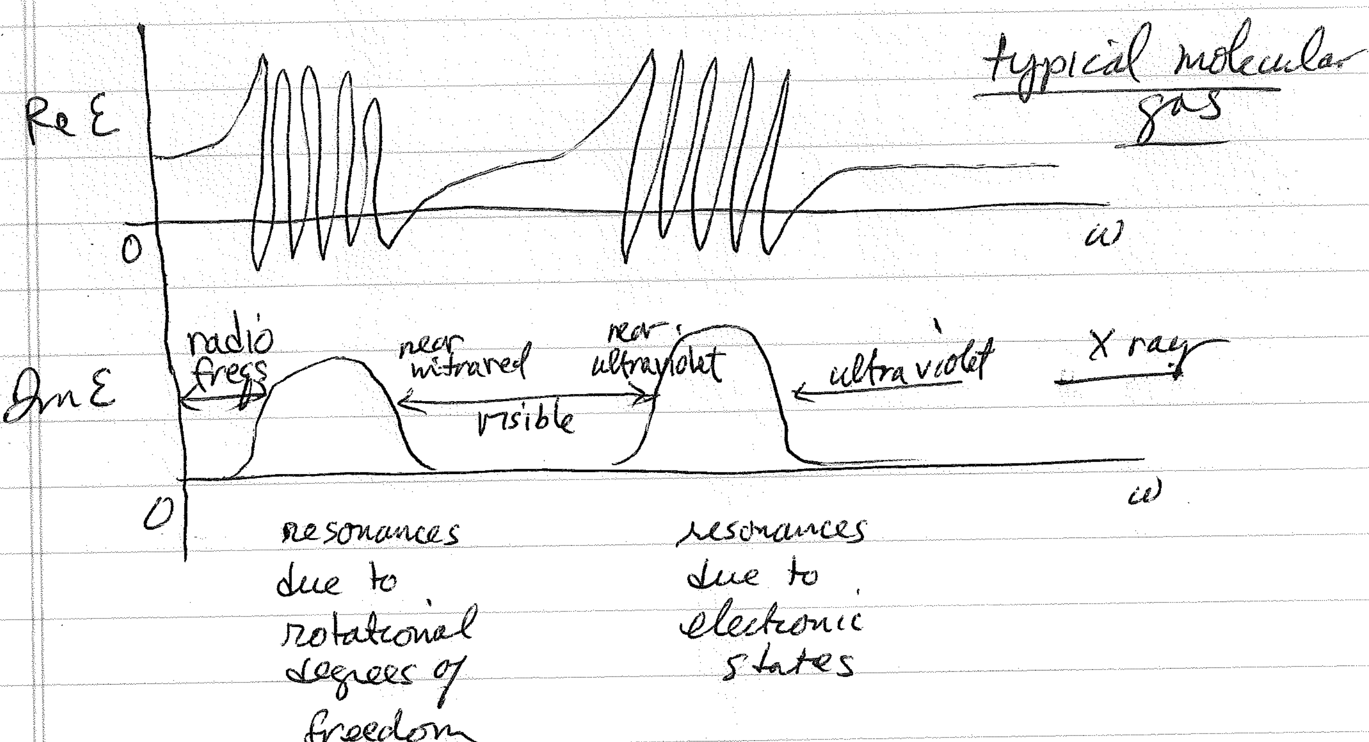
wave vector is almost pure imaginary
wave decays exponentially to zero in much less than one wavelength.

we will see this corresponds to total reflection
Since $\omega \gg \omega_0$, we are not at resonance so material is not absorbing much energy from wave. The strong attenuation is due to the destructive interference between the wave and the induced fields of the polarized atoms

Our simple model had a single resonance at ω_0
A more realistic model for molecules has many bands of resonances due to rotational, vibrational, and electronic modes of excitation

$$\varepsilon(\omega) = 1 + \omega_p^2 \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

where $\hbar\omega_i$ are spacings between energy levels with allowed electric dipole transitions



$$\omega_p = \sqrt{\frac{4\pi n e^2}{m}}$$

$$= 4.4\times10^{16}\sqrt{\frac{n}{n_A}}\ \text{sec}^{-1}, \quad n_A = 6\times10^{23}/\text{cm}^3$$

~~For H_2O $\frac{n}{n_A}$ ~ 0.05~~

$$\Rightarrow \hbar\omega_p = 185\sqrt{\frac{n}{n_A}}\ \text{ev}$$

For H_2O $\quad \frac{n}{n_A} \sim 0.05$

$$\hbar\omega_p \sim 40\ \text{ev}$$

For typical metal $\frac{n}{n_A} \sim 0.1$

$$\hbar\omega_p \sim 58\ \text{ev}$$

compared to $\hbar\omega_0 \sim$ ev