EM waves in Conductors

conduction electrons are mobil, not bound

$\Rightarrow$ we have to include the $\vec{J}_f$ and ρ_f from them.

Simple classical model for electron motion — "Drude" Model

$$m\ddot{\vec{r}} = -e\vec{E}(t) - \frac{m}{\tau}\dot{\vec{r}}$$

no restoring force!

external E field

damping force due to collisions

τ is "relaxation time"

$$\vec{E} = \vec{E}_\omega e^{-i\omega t} \Rightarrow \vec{r} = \vec{r}_\omega e^{-i\omega t} \text{ solution}$$

plug in to get

$$\left(-\omega^2 - \frac{i\omega}{\tau}\right)\vec{r}_\omega = -\frac{e}{m}\vec{E}_\omega \Rightarrow \vec{r}_\omega = \frac{e}{m}\frac{1}{\omega^2 + \frac{i\omega}{\tau}}\vec{E}_\omega$$

current is

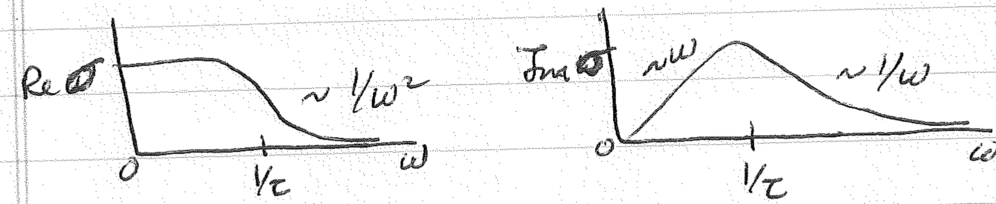
$$\vec{J}_f = -ne\dot{\vec{r}}, \quad \vec{J}_f = \vec{J}_\omega e^{-i\omega t} \Longrightarrow \vec{J}_\omega = -en(-i\omega)\vec{r}_\omega$$

density of electrons

$$\vec{J}_\omega = \frac{ne^2}{m}\frac{i\omega}{\omega^2 + \frac{i\omega}{\tau}}\vec{E}_\omega = \frac{ne^2\tau}{m}\frac{1}{1 - i\omega\tau}\vec{E}_\omega$$

$$\vec{J}_\omega = \sigma(\omega)\vec{E}_\omega$$

conductivity $\boxed{\sigma(\omega) = \frac{ne^2\tau}{m}\frac{1}{1 - i\omega\tau}}$



$$\mathrm{Re}\,\sigma = \frac{\sigma_0}{1+\omega^2\tau^2} \qquad \mathrm{Im}\,\sigma = \frac{\sigma_0\,\omega\tau}{1+\omega^2\tau^2}$$

$$\sigma_0 = \sigma(0) = \frac{ne^2\tau}{m}$$ dc conductivity

Charge density ρ_f given by charge conservation law.
For plane waves

$$\rho_f = \rho_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \quad \vec{J}_f = \vec{J}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\frac{\partial \rho_f}{\partial t} = -\vec{\nabla}\cdot\vec{J}_f \Rightarrow -i\omega\rho_\omega = -i\vec{k}\cdot\vec{J}_\omega$$

$$\rho_\omega = \frac{\vec{k}\cdot\vec{J}_\omega}{\omega} = \frac{\sigma(\omega)\,\vec{k}\cdot\vec{E}_\omega}{\omega}$$

<u>Maxwell Equations</u>

1) $\vec{\nabla}\cdot\vec{D} = 4\pi\rho_f$ 2) $\vec{\nabla}\cdot\vec{B} = 0$

3) $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ 4) $\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J}_f + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$

Assume $\vec{H} = \vec{B}/\mu$, μ constant

$\vec{D}_\omega = \varepsilon_b(\omega)\vec{E}_\omega$ $\varepsilon_b(\omega)$ is dielectric function from the bound charges

$\vec{J}_\omega = \sigma(\omega)\vec{E}_\omega$ $\sigma(\omega)$ is conductivity from free charges

$\rho_\omega = \frac{\sigma}{\omega}\vec{k}\cdot\vec{E}_\omega$

For harmonic plane wave solutions $\vec{E} = E_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ etc.

$$1) \Rightarrow \quad i\vec{k}\cdot\vec{D}_\omega = i\vec{k}\cdot\varepsilon_b \vec{E}_\omega = 4\pi \rho_\omega = 4\pi\sigma \frac{\vec{k}\cdot\vec{E}_\omega}{\omega}$$

$$\Rightarrow \quad i\vec{k}\cdot\vec{E}_\omega \left(\varepsilon_b + \frac{4\pi i\sigma}{\omega}\right) = 0$$

$$2) \Rightarrow \quad i\mu\vec{k}\cdot\vec{H}_\omega = 0$$

$$3) \Rightarrow \quad i\vec{k}\times\vec{E}_\omega = \frac{i\omega\vec{B}_\omega}{c} = \frac{i\omega\mu H_\omega}{c}$$

$$4) \Rightarrow \quad i\vec{k}\times\vec{H}_\omega = \frac{4\pi}{c}\vec{j}_\omega - \frac{i\omega}{c}\vec{D}_\omega$$

$$= \frac{4\pi}{c}\sigma\vec{E}_\omega - \frac{i\omega}{c}\varepsilon_b\vec{E}_\omega$$

$$= -\frac{i\omega}{c}\left(\varepsilon_b + \frac{4\pi i\sigma}{\omega}\right)\vec{E}_\omega$$

Notice: all the equations above look exactly like what we had for the dielectric, provided we define

$$\boxed{\varepsilon(\omega) = \varepsilon_b(\omega) + \frac{4\pi i\sigma(\omega)}{\omega}}$$

So all results for the dielectric case carry over to conductors, provided we make the above substitution. In particular

dispersion relation for transverse modes $\boxed{k^2 = \frac{\omega^2}{c^2}\mu\,\varepsilon(\omega)}$

The main difference between dielectrics & conductors has to do with the contribution that the $4\pi i\sigma/\omega$ makes to the real and imaginary parts of $\varepsilon(\omega)$.

For simple Drude model $\sigma(\omega) = \dfrac{\sigma_0}{1-i\omega\tau}$ $\sigma_0 = \dfrac{ne^2\tau}{m}$

① <u>Low frequencies $\omega \ll 1/\tau$</u>, $\omega \ll \omega_0$ ← resonant freq of bound electrons

$\varepsilon_b(\omega) \approx \varepsilon_b(0)$ real

$\sigma(\omega) \approx \sigma_0$ real

$$\Rightarrow \boxed{\varepsilon(\omega) \approx \varepsilon_b(0) + \frac{4\pi i\sigma_0}{\omega}}$$
← gives large ε_2 as $\omega \to 0$
⇒ strong dissipation

$\mathrm{Re}\,\varepsilon = \varepsilon_1$

$\mathrm{Im}\,\varepsilon = \varepsilon_2$

when $\dfrac{\varepsilon_2}{\varepsilon_1} = \dfrac{4\pi\sigma_0}{\omega\varepsilon_b(0)} \gg 1$ we call this regime a "<u>good</u>" conductor.
conduction electrons dominate the response
– waves strongly attenuated

when $\dfrac{\varepsilon_2}{\varepsilon_1} = \dfrac{4\pi\sigma_0}{\omega\,\varepsilon_b(0)} \ll 1$ we call this regime a "<u>poor</u>" conductor.
little absorption of energy by conduction electrons.
waves propagate

One always enters the "good" conductor region when ω gets sufficiently small.

wave vector:

$$k = \frac{\omega}{c}\sqrt{\mu\varepsilon}$$

for a good conducter where $\varepsilon_2 \gg \varepsilon_1$,

$$\varepsilon \sim i\varepsilon_2 = \frac{4\pi i \sigma_0}{\omega}$$

$$k = k_1 + i k_2 = \frac{\omega}{c}\sqrt{\mu\,\frac{4\pi i \sigma_0}{\omega}} \qquad \sqrt{i} = \frac{1+i}{\sqrt{2}}$$

$$k_1 = k_2 = \frac{\omega}{c}\sqrt{\frac{4\pi\mu\sigma_0}{2\omega}} = \frac{1}{c}\sqrt{2\pi\mu\sigma_0\omega}$$

for $\vec{k} = k\hat{z}$,

$$\vec{E} = \vec{E}_\omega e^{i(kz-\omega t)} = \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$$

$$\delta \equiv 1/k_2 = \frac{c}{\sqrt{2\pi\mu\sigma_0\omega}}$$

"skin depth" distance wave propagates into conductor

$\delta \sim 1/\sqrt{\omega}$ increases as ω decreases

ϕ phase shift between oscillations of $\vec{E}$ and $\vec{H}$

$$\phi = \arctan(k_2/k_1) \simeq \arctan(1) = 45°$$

Amplitude ratio

$$\frac{|\vec{H}_\omega|}{|\vec{E}_\omega|} = \frac{c|k|}{\omega\mu} = \frac{\sqrt{2}\,c}{\omega\mu}k_1$$

$$= \frac{\sqrt{2}\,c}{\omega\mu}\frac{1}{c}\sqrt{2\pi\mu\sigma_0\omega}$$

$$= \sqrt{\frac{4\pi\sigma_0}{\omega\mu}} \sim 1/\sqrt{\omega}$$

as $\omega \to 0$, most of the energy of the wave is carried by the magnetic field part

(2) <u>high frequencies</u> $\omega \gg 1/\tau$, $\omega \gg \omega_0$

$\varepsilon_b(\omega) \approx 1$

$$\sigma(\omega) \approx \frac{\sigma_0}{-i\omega\tau} = \frac{i n e^2 \tau}{m\omega\tau} = \frac{i n e^2}{m\omega}$$ pure imaginary indep of τ

$$\varepsilon(\omega) \simeq 1 + \frac{4\pi i \sigma}{\omega} \simeq 1 - \frac{4\pi n e^2}{m\omega^2}$$

$$\boxed{\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}}$$ $\omega_p \equiv \sqrt{\frac{4\pi n e^2}{m}}$

plasma freq of the conduction electrons

$\varepsilon(\omega)$ is <u>real</u>

1) If $\omega > \omega_p$ then $\varepsilon > 0$

$\Rightarrow$ transparent propagation

$k = k_1 = \frac{\omega}{c}\sqrt{\mu\varepsilon}$ is pure real

$k_2 \approx 0$

2) If $\omega < \omega_p$ then $\varepsilon < 0$

$\Rightarrow$ total reflection

$k_1 \approx 0$ $\quad$ k is pure imaginary

$k = k_2 = \frac{\omega}{c}\sqrt{\mu|\varepsilon|}$

ω_p gives cross over between total reflection and transparent propogation for typical metals

$$\tau \sim 10^{-14} \text{sec}$$
$$\omega_p \sim 10^{16} \text{sec}^{-1}$$
$$\lambda_p = \frac{2\pi c}{\omega_p} \sim 3 \times 10^3 \text{Å} \quad (\text{visible is } \lambda \sim 5 \times 10^3 \text{Å})$$

<u>Example</u>: The ionesphere is a layer of charged gas surrounding the earth. In many respects the charged particles of the ionesphere behave like conduction electrons in a metal. The plasma freq of the ionesphere is such that

for AM radio $\omega_{AM} < \omega_p \Rightarrow$ AM radio signals reflected back to earth

for FM radio $\omega_{FM} > \omega_p \Rightarrow$ FM radio signals propagate through ionesphere into space

Explains why you can pick up AM stations from far away - they get reflected back

But you can only pick up local FM stations.

Longitudinal modes in conductors

ie $\vec{H}_\omega$ or $\vec{E}_\omega$ not $\perp \vec{k}$

magnetic field

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow i\mu\vec{k}\cdot\vec{H}_\omega = 0 \Rightarrow \vec{H}_\omega \perp \vec{k}$ transverse

or $\vec{k} = 0$ spatially uniform $\vec{H}$

if $\vec{k} = 0$ then Faraday

$$\underbrace{i\vec{k}\times\vec{E}_\omega}_{=0 \text{ as } \vec{k}=0} = \frac{i\omega\mu}{c}\vec{H}_\omega = 0 \Rightarrow \omega = 0$$

So only possible longitudinal $\vec{H}$ is spatially uniform, constant in time.

electric field

$\vec{\nabla}\cdot\vec{D} = 4\pi\rho_f \Rightarrow i\varepsilon(\omega)\vec{k}\cdot\vec{E}_\omega = 0 \Rightarrow \vec{E}_\omega \perp \vec{k}$ transverse

or $\varepsilon(\omega) = 0$

If $\vec{E}_\omega \parallel \vec{k}$ but $\varepsilon(\omega) = 0$, then can satisfy all other maxwell equations.

$$i\vec{k}\times\vec{E}_\omega = \frac{i\omega\mu}{c}\vec{H}_\omega \Rightarrow \vec{H}_\omega = 0$$

$$\Rightarrow i\mu_0\vec{k}\cdot\vec{H}_\omega = 0 \quad \text{and} \quad \underbrace{i\vec{k}\times\vec{H}_\omega}_{=0 \text{ as } \vec{H}_\omega=0} = \underbrace{-\frac{i\omega}{c}\varepsilon(\omega)\vec{E}_\omega}_{=0 \text{ as } \varepsilon(\omega)=0}$$

So we can have longitudinal electric field oscillation when $\varepsilon(\omega) = 0$

<u>low freq</u> $\quad \omega \ll \omega_0 \quad \omega\tau \ll 1$

$$\varepsilon \simeq \varepsilon_b(0) + \frac{4\pi i \sigma_0}{\omega}$$

$$\varepsilon(\omega) = 0 \quad \text{when} \quad \omega = -\frac{4\pi i \sigma_0}{\varepsilon_b(0)}$$

$$\vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \vec{E}_\omega e^{-\frac{4\pi\sigma_0}{\varepsilon_b(0)}t} e^{i\vec{k}\cdot\vec{r}}$$

If set up a longitudinal $\vec{E}$ field, it decays to zero exponentially with ~~time const~~ decay time $\varepsilon_b(0)/4\pi\sigma_0$
This is consistant with assumption the $\vec{E}=0$ inside a conductor for electrostatics

in statics $\vec{E} = -\vec{\nabla}\phi \Rightarrow \vec{E} \sim -i\vec{k}\phi_k e^{i\vec{k}\cdot\vec{r}}$ is longitudinal

<u>high freq</u> $\quad \omega \gg 1/\tau, \quad \omega \gg \omega_0$

$$\varepsilon(\omega) \approx 1 + \frac{4\pi i\sigma}{\omega} = 1 - \frac{\omega_p^2}{\omega^2} \qquad \omega_p^2 = \frac{4\pi n e^2}{m}$$

$$\varepsilon = 0 \quad \text{when} \quad \omega = \omega_p$$

So we have oscillatory longitudinal $\vec{E}$ only when $\omega = \omega_p$, independent of $\vec{k}$.

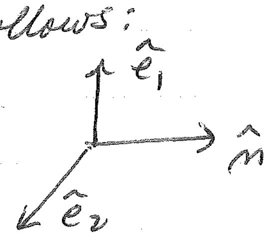
$$\vec{E} = \vec{E}_\omega e^{i\vec{k}\cdot\vec{r}} e^{-i\omega_p t}$$

This is called a plasma oscillation. When one quantizes this oscillatory mode, it is called a <u>plasmon</u>

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \Rightarrow \rho = \frac{i\vec{k}\cdot\vec{E}_\omega}{4\pi} e^{i\vec{k}\cdot\vec{r}} e^{-i\omega_p t}$$ ← plasma osc. is a charge density oscillation

Polarization

Consider a transverse plane wave traveling in direction $\hat{n}$, ie $\vec{k} = k\hat{n}$. Define a right handed coordinate system as follows:



$$\hat{e}_1 \times \hat{e}_2 = \hat{n}$$
$$\hat{n} \times \hat{e}_1 = \hat{e}_2$$
$$\hat{e}_2 \times \hat{n} = \hat{e}_1$$

A general solution to Maxwell's equations for a transverse plane wave is then

$$\vec{E}(\vec{r},t) = Re\left\{(E_1\hat{e}_1 + E_2\hat{e}_2)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right\}$$
$$\vec{H}(\vec{r},t) = \frac{c}{\omega\mu} Re\left\{k\,\hat{n}\times(E_1\hat{e}_1 + E_2\hat{e}_2)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right\}$$
$$= \frac{c}{\omega\mu} Re\left\{k\,(E_1\hat{e}_2 - E_2\hat{e}_1)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right\}$$

In general, k is complex

$$k = k_1 + i k_2 = |k|\, e^{i\delta}, \quad \begin{cases} |k| = \sqrt{k_1^2 + k_2^2} \\ \delta = \arctan(k_2/k_1) \end{cases}$$

So far we implicitly assumed that E_1 and E_2 are real constants. In this case

$$\vec{E}(\vec{r},t) = \vec{E}_\omega\, e^{-k_2\hat{n}\cdot\vec{r}} \cos(k_1\hat{n}\cdot\vec{r} - \omega t)$$
$$\vec{H}(\vec{r},t) = \vec{H}_\omega\, e^{-k_2\hat{n}\cdot\vec{r}} \cos(k_1\hat{n}\cdot\vec{r} - \omega t + \delta)$$

where

$$\vec{E}_\omega \equiv E_1\hat{e}_1 + E_2\hat{e}_2 \quad \text{and} \quad \vec{H}_\omega \equiv \frac{c|k|}{\omega\mu}(E_1\hat{e}_2 - E_2\hat{e}_1)$$

are fixed vectors for all time and space.