

Kramers - Kronig Relation

We saw that $\vec{P}_\omega = \alpha(\omega) \vec{E}_\omega \Rightarrow \vec{P}(t) = \int_{-\infty}^{\infty} \alpha(t-t') \vec{E}(t')$

Causal response is $\tilde{\alpha}(t) = 0$ for $t < 0$

$\Rightarrow \alpha(\omega)$ has no poles in upper half of complex ω plane (UHP)

For any complex $\bar{\omega}$ in upper half of complex ω plane,

$$\alpha(\bar{\omega}) = \frac{1}{2\pi i} \oint \frac{\alpha(\omega')}{\omega' - \bar{\omega}} d\omega' \quad \text{since no poles of } \alpha \text{ in UHP}$$



contour along real axis, closed

at infinity in UHP. The closing

semicircle at infinity gives

no contribution assuming $\alpha(\omega)$

decays quickly enough as $|\omega| \rightarrow \infty$

only pole of
integrand is
at $\omega' = \bar{\omega}$

$$\Rightarrow \alpha(\bar{\omega}) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega' - \bar{\omega}}$$

Now consider $\bar{\omega} = \omega + i\delta$ where ω and δ are real

and $\delta \rightarrow 0$

$$\alpha(\omega) = \lim_{\delta \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega' - \omega - i\delta}$$

$$\text{Now } \frac{1}{\omega' - \omega - i\delta} = P\left(\frac{1}{\omega' - \omega}\right) + i\pi\delta(\omega' - \omega)$$

\Uparrow principle part

$$\Rightarrow \alpha(\omega) = \frac{1}{\pi i} \mathcal{P} \int_{-\infty}^{\infty} \frac{\alpha(\omega') d\omega'}{\omega' - \omega}$$

$$\Rightarrow \left. \begin{aligned} \operatorname{Re} \alpha(\omega) &= \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\operatorname{Im} \alpha(\omega') d\omega'}{\omega' - \omega} \\ \operatorname{Im} \alpha(\omega) &= -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\operatorname{Re} \alpha(\omega') d\omega'}{\omega' - \omega} \end{aligned} \right\} \begin{array}{l} \text{Kramer} \\ \text{Kronig} \\ \text{relations} \end{array}$$

If know Re α or Im α can reconstruct full complex α

True for any causal response function

details:

$$\begin{aligned} \alpha(\omega) &= \frac{1}{2\pi i} \mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega' - \omega} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' i\pi \delta(\omega' - \omega) \alpha(\omega) \\ &= \frac{1}{2\pi i} \mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega' - \omega} + \frac{1}{2} \alpha(\omega) \end{aligned}$$

$$\Rightarrow \frac{1}{2} \alpha(\omega) = \frac{1}{2\pi i} \mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega' - \omega}$$

$$\text{or } \alpha(\omega) = \frac{1}{\pi i} \int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega' - \omega} \quad \text{as above}$$

Radiation From moving Charges

In Lorentz gauge $\frac{1}{c} \frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0$

$$\left. \begin{aligned} \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} &= -4\pi \rho \\ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} &= -\frac{4\pi}{c} \vec{j} \end{aligned} \right\} \begin{array}{l} \text{wave equation} \\ \text{with source} \end{array}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{\nabla} \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

If we can solve wave equation with source (inhomogeneous wave equation) then we are in principle done! To do this we want to find the Green's function for the wave equation

Recall from statics: $\nabla^2 \phi = -4\pi \rho$

Green's function satisfies $\nabla^2 G(\vec{r}) = -4\pi \delta(\vec{r})$

Then $\phi(\vec{r}) = \int d^3 r' G(\vec{r} - \vec{r}') \rho(\vec{r}') + \phi_0$

solution for infinite volume that vanishes as $r \rightarrow \infty$ is

$$G(\vec{r} - \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|}$$

$$\nabla^2 \phi_0 = 0$$

For wave equation we want solution to

$$\nabla^2 G(\vec{r}, t; \vec{r}', t') - \frac{1}{c^2} \frac{\partial^2 G(\vec{r}, t; \vec{r}', t')}{\partial t^2} = -4\pi \delta(\vec{r} - \vec{r}') \delta(t - t')$$

$$\text{Then we will have } \left\{ \begin{array}{l} \phi(\vec{r}, t) = \int dt' \int d^3 r' G(\vec{r}, t; \vec{r}', t') \rho(\vec{r}', t') + \phi_0 \\ \vec{A}(\vec{r}, t) = \frac{1}{c} \int dt' \int d^3 r' G(\vec{r}, t; \vec{r}', t') \vec{j}(\vec{r}', t') + \vec{A}_0 \end{array} \right.$$

where $\nabla^2 \phi_0 - \frac{1}{c^2} \frac{\partial^2 \phi_0}{\partial t^2} = 0$ similarly for \vec{A}_0

ϕ_0 or \vec{A}_0 could describe an incoming wave for example
 To construct the Green's function.

For infinite space (but not, for example, inside a cavity)

$$G(\vec{r}_t; \vec{r}'_t) = G(\vec{r} - \vec{r}', t - t')$$

express as Fourier transform

$$G(\vec{r}, t) = \int \frac{d^3k d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) G(\vec{r}, t) = \int \frac{d^3k d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega) \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\equiv \int \frac{d^3k d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega) \left[-k^2 + \frac{\omega^2}{c^2}\right] e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$-4\pi \delta(\vec{r}) \delta(t) = -4\pi \int \frac{d^3k d\omega}{(2\pi)^4} e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

equate Fourier amplitudes

$$\Rightarrow \left[-k^2 + \frac{\omega^2}{c^2}\right] \tilde{G}(\vec{k}, \omega) = -4\pi$$

$$\boxed{\tilde{G}(\vec{k}, \omega) = \frac{4\pi c^2}{k^2 c^2 - \omega^2}}$$

when $\omega^2 \neq c^2 k^2$

$$G(\vec{r}, t) = \int \frac{d^3k d\omega}{(2\pi)^4} \frac{4\pi c^2}{k^2 c^2 - \omega^2} e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

↑ poles at $\omega = \pm ck$

In evaluating the ω integral we have to know how

to treat the poles on the real axis so that $G(\vec{r}, t)$ will have the desired behavior.

What we want is for $G(\vec{r}, t)$ to be causal, i.e. $G(\vec{r}, t) = 0$ for $t < 0$, so $\phi(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$ depend only on the values of the sources at earlier times $t' < t$.

$$\int d^3k e^{i\vec{k}\cdot\vec{r}} \tilde{G}(\vec{k}, \omega) = 2\pi \int_0^\pi d\theta \sin\theta \int_0^\infty dk k^2 e^{ikr \cos\theta} \tilde{G}(k, \omega)$$

$$= 2\pi \int_{-1}^1 d\mu \int_0^\infty dk k^2 e^{ikr\mu} \tilde{G}(k, \omega)$$

$$= 4\pi \int_0^\infty dk k^2 \frac{\sin kr}{kr} \tilde{G}(k, \omega)$$

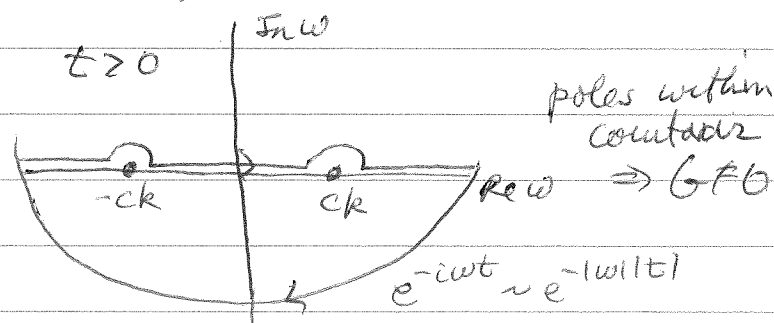
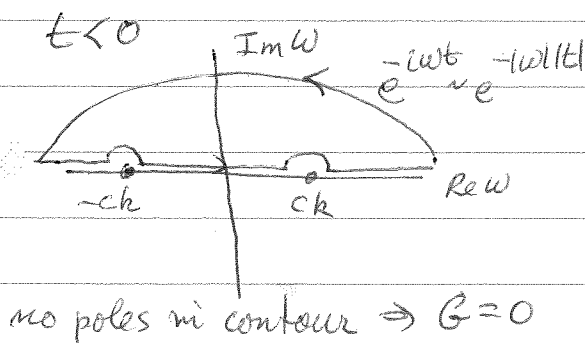
$\mu = \cos\theta$
 $d\mu = -d\theta \sin\theta$

$\vec{k} = (k, \theta, \varphi)$
in spherical coordinates

$$G(\vec{r}, t) = -\frac{c^2}{4\pi^2} \int_0^\infty dk k^2 \frac{\sin kr}{kr} \int_C \frac{e^{-z\omega t}}{(\omega + ck)(\omega - ck)} d\omega$$

contour along real axis, but deformed to go around the poles

for $t < 0$, $e^{-z\omega t}$ will decay exponentially fast for large $|\omega|$ in the upper half complex (UHP) ω plane \Rightarrow can close contour in UHP for $t < 0$. If we want $G = 0$ for $t < 0$, there should therefore be no poles in UHP. The contour C we want is therefore:



with this convention for the contour C we can evaluate the ω -integral using Cauchy's residue theorem

$$\text{For } t > 0, \quad \int \frac{e^{-i\omega t}}{(\omega + ck)(\omega - ck)} d\omega = -2\pi i \left[\frac{e^{-ickt}}{2ck} - \frac{e^{ickt}}{2ck} \right]$$

$$= -\frac{2\pi \sin(ckt)}{ck}$$

$$G(\vec{r}, t) = \frac{2c}{\pi r} \int_0^\infty dk \sin(kr) \sin(ckt) = \frac{c}{\pi r} \int_{-\infty}^\infty dk \frac{e^{ikr} - e^{-ikr}}{(-4)} (e^{ickt} - e^{-ickt})$$

$$= -\frac{c}{2r} \int_{-\infty}^\infty \frac{dk}{2\pi} \left[e^{i(r+ct)k} + e^{-i(r+ct)k} - e^{i(r-ct)k} - e^{-i(r-ct)k} \right]$$

each integral would give a δ -function, but for 1st two terms $\delta(r+ct) = 0$ since here $t > 0$ (by definition) and $r = |\vec{r}| \geq 0$ so the argument will never vanish.

$$\boxed{G(\vec{r}, t) = \frac{c}{r} \delta(r-ct) = \frac{\delta(t - r/c)}{r}} \quad \text{using } \delta(ax) = \frac{1}{|a|} \delta(x)$$

$$G(\vec{r}, t; \vec{r}', t') = \begin{cases} \frac{\delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|} & t-t' \geq 0 \\ 0 & t-t' < 0 \end{cases} \quad \left. \begin{array}{l} \text{Green's function} \\ \text{for wave equation} \\ \text{in free space} \end{array} \right\}$$

$G \neq 0$ only on "light cone" that emanates from (\vec{r}', t') , i.e. when $|\vec{r}-\vec{r}'| = c(t-t')$.
Signal from source at (\vec{r}', t') travels with C .

$$\phi(\vec{r}, t) = \phi_0(\vec{r}, t) + \int_{-\infty}^t d^3r' \int dt' \frac{\delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|} \rho(\vec{r}', t')$$

$$\vec{A}(\vec{r}, t) = \vec{A}_0(\vec{r}, t) + \int_{-\infty}^t d^3r' \int dt' \frac{\delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|} \vec{j}(\vec{r}', t')$$

Apply to a single moving point charge

$$\rho(\vec{r}, t) = q \delta(\vec{r} - \vec{r}_0(t))$$

$$\vec{j}(\vec{r}, t) = q \vec{v}(t) \delta(\vec{r} - \vec{r}_0(t)) \quad \text{where } \vec{v}(t) = \frac{d\vec{r}_0}{dt}$$

Then

$$\phi(\vec{r}, t) = q \int dt' \frac{\delta(t-t' - \frac{1}{c} |\vec{r} - \vec{r}_0(t')|)}{|\vec{r} - \vec{r}_0(t')|}$$

because of the $\vec{r}_0(t')$ in the argument of the $\delta()$ function the t' dependence is not of the simple form $t' - t_0$.

We can write

$$g(t') \equiv t' + \frac{1}{c} |\vec{r} - \vec{r}_0(t')|$$

then

$$\phi(\vec{r}, t) = q \int dt' \frac{\delta(t - g(t'))}{|\vec{r} - \vec{r}_0(t')|}$$

$$= q \int \frac{\delta(t - g(t'))}{|\vec{r} - \vec{r}_0(t')|} dg \left(\frac{dt'}{dg} \right)$$

$$= \frac{q}{|\vec{r} - \vec{r}_0(t')|} \frac{1}{(dg/dt')} \Bigg|_{t' \text{ such that } g(t') = t}$$

$$g(t') = t' + \frac{1}{c} \sqrt{[x-x_0(t')]^2 + [y-y_0(t')]^2 + [z-z_0(t')]^2}$$

$$\frac{dg}{dt'} = 1 + \frac{1}{c|\vec{r}-\vec{r}_0(t')|} \left\{ [x-x_0(t')] \left(-\frac{dx_0}{dt'} \right) + \dots \right\}$$

$$= 1 - \frac{1}{c} \hat{n}(t') \cdot \vec{v}(t')$$

where $\hat{n}(t') = \frac{\vec{r}-\vec{r}_0(t')}{|\vec{r}-\vec{r}_0(t')|}$ unit vector pointing from $\vec{r}_0(t')$ to \vec{r}

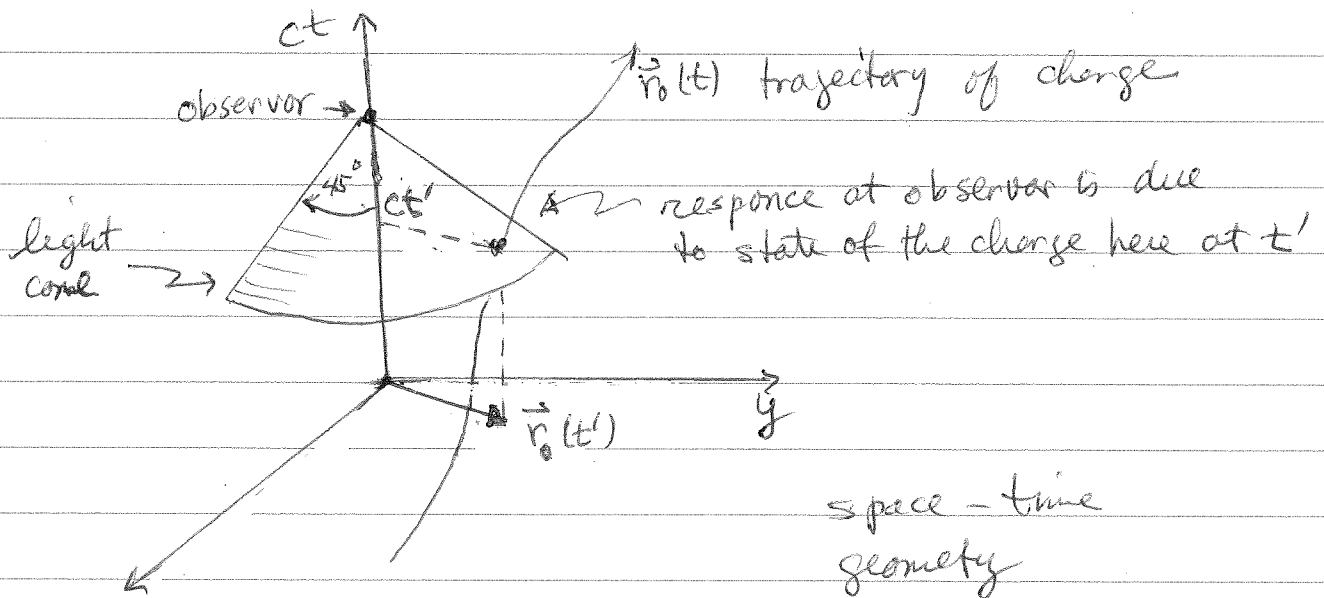
$$\phi(\vec{r}, t) = \frac{q}{|\vec{r}-\vec{r}_0(t')| \left[1 - \frac{1}{c} \hat{n}(t') \cdot \vec{v}(t') \right]}$$

$$\vec{A}(\vec{r}, t) = \frac{q \vec{v}(t')/c}{|\vec{r}-\vec{r}_0(t')| \left[1 - \frac{1}{c} \hat{n}(t') \cdot \vec{v}(t') \right]}$$

Liénard
-
Wiechert
Potentials

where t' is determined by the condition

$$t - t' = \frac{1}{c} |\vec{r} - \vec{r}_0(t')|$$



For charge moving with constant velocity along \hat{z}

$$\vec{r}_0(t) = vt \hat{z} \quad \vec{v} = \frac{d\vec{r}_0}{dt} = v \hat{z}$$

For observer at position \vec{r} , time t , the fields will be determined by the charge at time t' such that

$$t - t' - \frac{|\vec{r} - \vec{r}_0(t')|}{c} = 0$$

$$t - t' - \frac{\sqrt{r^2 + v^2 t'^2}}{c} = 0$$

$$(t - t')^2 = \frac{r^2 + v^2 t'^2}{c^2}$$

$$(1 - v^2/c^2) t'^2 - 2tt' + t^2 - \frac{r^2}{c^2} = 0$$

$$\text{let } \gamma = (1 - v^2/c^2)^{-1/2}$$

$$t'^2 - 2\gamma^2 t t' + \gamma^2 (c^2 t^2 - r^2) = 0$$

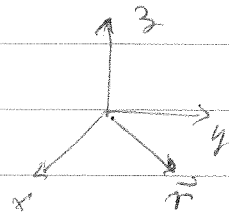
$$t' = \gamma^2 t \pm \sqrt{\gamma^4 t^2 - \gamma^2 t^2 + \gamma^2 \frac{r^2}{c^2}}$$

$$= \gamma^2 t \pm \sqrt{\gamma^2 (\gamma^2 t^2 - t^2 + \frac{r^2}{c^2})}$$

$$\gamma^2 - 1 = \frac{1}{1 - v^2/c^2} - 1 = \frac{v^2/c^2}{1 - v^2/c^2} = \gamma^2 \frac{v^2}{c^2}$$

$$= \gamma^2 t \pm \gamma \sqrt{t^2 \gamma^2 \left(\frac{v^2}{c^2} \right) + \frac{r^2}{c^2}}$$

$$t' = \gamma^2 t \pm \frac{\gamma}{c} \sqrt{v^2 t^2 + r^2}$$



consider $t=0$. solution should give $t' < 0$
 $\Rightarrow (-)$ sign is the solution we want

$$t' = \gamma^2 t - \frac{\gamma^2}{c} \sqrt{v^2 t^2 + \frac{r^2}{\gamma^2}}$$

$$\phi(\vec{r}, t) = \frac{q}{|\vec{r} - \vec{r}_0(t')| \left[1 - \frac{1}{c} \hat{M}(t') \cdot \vec{v} \right]}$$

~~$$|\vec{r} - \vec{r}_0(t')| = \sqrt{r^2 + v^2 t'^2} = c(t - t')$$~~

$$|\vec{r} - \vec{r}_0(t')| = \sqrt{r^2 + v^2 t'^2} = c(t - t') \quad \leftarrow \text{from condition that determines } t'$$

$$(\vec{r} - \vec{r}_0(t')) \cdot \vec{v} = -\vec{r}_0(t') \cdot \vec{v} \quad \text{for } \vec{v} = v \hat{z}$$

$$= -v^2 t' \quad \vec{r} \text{ in } xy \text{ plane}$$

$$\phi(\vec{r}, t) = \frac{q}{c(t - t') \left[1 + \frac{v^2 t'}{c^2(t - t')} \right]}$$

$$= \frac{q}{c(t - t') + \frac{v^2 t'}{c}} = \frac{q}{c \left[t - \left(1 - \frac{v^2}{c^2}\right) t' \right]}$$

$$= \frac{q}{c \left(t - \frac{t'}{\gamma^2} \right)} = \frac{q}{c \frac{1}{\gamma^2} \sqrt{v^2 t^2 + \frac{r^2}{\gamma^2}}}$$

$$\phi(\vec{r}, t) = \frac{q}{\sqrt{v^2 t^2 + \frac{r^2}{\gamma^2}}}$$

$$\vec{A}(\vec{r}, t) = \frac{q \vec{v}}{c \sqrt{v^2 t^2 + \frac{r^2}{\gamma^2}}}$$

solutions for
 \vec{r} in xy plane
 when charge passes
 through xy plane
 at $t=0$