Special Relativity

1) Speed of light is constant in all inertial frames of reference.
2) Physical laws must look the same in all inertial frames of reference — there is no experiment that can determine the "absolute" velocity of any inertial frame.

⇒ If a flash of light goes off at the origin of some coord system, the outgoing wavefronts look spherical in all inertial frames.
   Equation of wavefront in $K$: $t^2 - c^2x^2 = 0$

⇒ $(x,y,z,t)$ coords in one inertial frame $K$
⇒ $(x',y',z',t')$ coords in another inertial frame $K'$ that moves with velocity $\vec{v} = v\hat{x}$ with respect to $K$.

What is the transformation that relates coords in $K'$ to coords in $K$:

$y = y' \quad z = z'$

⇒ $c^2t^2 - x^2 = c^2t'^2 - x'^2$

⇒ $\frac{(ct+x)}{(ct'+x')} = \frac{(ct-x)}{(ct'-x')}$

Expect transformation to be linear.

⇒ $ct' + x' = (ct + x)f$
⇒ $ct' - x' = (ct - x)f^{-1}$

for some constant $f$. Write $f = e^{-y}$, $y$ is rapidity.
Solve for $ct'$ and $x'$ in terms of $ct$ and $x$.

\[ ct' = ct \left( \frac{e^y + e^{-y}}{2} \right) = x \left( \frac{e^y - e^{-y}}{2} \right) \]

\[ x' = -ct \left( \frac{e^y - e^{-y}}{2} \right) + x \left( \frac{e^y + e^{-y}}{2} \right) \]

\[ ct' = ct \cosh y - x \sinh y \]

\[ x' = -ct \sinh y + x \cosh y \]

**Meaning of parameter $y$**

\[ (ct \to 0) \]

The origin of $K'$ has trajectory $x' = -ct'$ in $K'$.

\[ \Rightarrow \frac{x'}{ct'} = -\frac{ct}{ct} \]

From transformation above, with $x = 0$, we get

\[ \frac{x'}{ct'} = -ct \sinh y \]

\[ \frac{ct'}{ct} \cosh y \]

So

\[ \frac{ct}{ct'} = \tanh y \]

\[ \Rightarrow \cosh y = \frac{1}{\sqrt{1 - \left(\frac{ct}{ct'}\right)^2}} \equiv \gamma \]

\[ \sinh y = \frac{ct}{ct'} \gamma \]

Lorentz Transformation:

\[ \begin{cases} ct' = \gamma ct - \gamma \left(\frac{ct}{c}\right) x \\ x' = -\gamma \left(\frac{ct}{c}\right) ct + \gamma x \end{cases} \]
Inverse transform obtained by taking $v \to -v$ in above
\[
\begin{align*}
ct &= \gamma ct' + \gamma \frac{v}{c} x' \\
x &= \gamma \frac{v}{c} ct' + \gamma x'
\end{align*}
\]

4-vectors

\[4-\text{position: } x'_\mu = (x'_1, x'_2, x'_3, \gamma ct') \quad x'_4 = ct \]
\[x'_\mu x'_\nu = \sum_{\mu=1}^{3} x'_\mu^2 = r^2 - ct'^2 \quad \text{lorentz invariant scalar}
\]

- has same value in all

Lorentz transform

\[
\begin{align*}
x'_1 &= \gamma \left( x_1 + \frac{v}{c} x_4 \right) \\
x'_2 &= x_2 \\
x'_3 &= x_3 \\
x'_4 &= \gamma \left( x_4 - \frac{v}{c} x_1 \right)
\end{align*}
\]

or \[x'_\mu = a_{\mu\nu} (L) x_\nu \]

\[a(L) = \begin{pmatrix}
\gamma & 0 & 0 & \frac{v}{c}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
-\frac{v}{c} \gamma & 0 & 0 & \gamma
\end{pmatrix}
\]

Inverse: \[x_\mu = a_{\mu\nu} (L^{-1}) x'_\nu \]

\[a_{\mu\nu} (L^{-1}) \text{ is given by taking } v \to -v \text{ in } a_{\mu\nu} (L)\]

we see \[a_{\mu\nu} (L^{-1}) = a_{\nu\mu} (L)\]

inverse = transpose
More generally

Since \( x^\mu \) is Lorentz moment scalar,

\[
x^\mu \cdot x^\nu = g_{\mu\nu} x^\mu x^\nu = x^2
\]

\[
\Rightarrow a^\mu_\nu l^\nu c_\mu l = x^2
\]

\[
\Rightarrow a^\mu_\nu l = g_{\mu\nu} = \delta_{\mu\nu}
\]

\[
\Rightarrow a_{\mu
u} = a_{\nu\mu} \text{ transpose = inverse}
\]

\[
a_{\mu\nu} \text{ is } 4 \times 4 \text{ orthogonal matrix}
\]

If \( L_1 \) is a Lorentz transform from \( K \) to \( K' \)

\[
L_2 \text{ is a Lorentz transform from } K' \text{ to } K''
\]

Then the Lorentz transform from \( K \) to \( K'' \) is given by the matrix

\[
a(L_2 L_1) = a(L_2) a(L_1)
\]

\[
\text{if } L_1 = L_2 \text{ and } L_2 = L_2^* \text{ so } L_2 L_1 = I \text{ identity}
\]

\[
\Rightarrow a^{-1}(L_2) = a(L_2^*)
\]

\[
dx^\mu = (dx_1, dx_2, dx_3, i c dt)
\]

\[
-(dx^\mu)^2 = c^2 ds^2 = c^2 dt^2 - dr^2 \text{ Lorentz moment scalar}
\]

\[
ds^2 = \frac{dt^2}{c^2} \left[ 1 - \frac{1}{c^2} \left( \frac{dx_1}{dt} \right)^2 - \frac{1}{c^2} \left( \frac{dx_2}{dt} \right)^2 - \frac{1}{c^2} \left( \frac{dx_3}{dt} \right)^2 \right]
\]

\[
ds^2 = \frac{dt^2}{c^2}
\]

\[
ds = \frac{dt}{c} \text{ proper time interval}
\]
A 4-vector is any 4 numbers that transform under a Lorentz transformation the same way as does $x_μ$

$\text{4-velocity} \quad u_μ = \frac{dx_μ}{ds} = \gamma \frac{dx_μ}{dt}$

space component $\vec{u} = \gamma \vec{v}$

$u_4 = i c \gamma$

$u_μ u_μ = \gamma^2 v^2 - c^2 \gamma^2 = \gamma^2 \left( v^2 - c^2 \right)$

$= \frac{v^2 - c^2}{1 - \frac{v^2}{c^2}} = -c^2$

$\text{4-acceleration} \quad \alpha_μ = \frac{du_μ}{ds} = \gamma \frac{du_μ}{dt}$

$\text{4-gradient} \quad \frac{\partial}{\partial x_μ} = \left( \frac{\partial}{\partial \gamma}, -i \frac{\partial}{\partial \gamma} \right)$

$\text{proof} \quad \frac{\partial}{\partial x_μ}$ is a 4-vector

$\frac{\partial}{\partial x_μ} x_μ = \frac{\partial}{\partial x_μ} x_μ = \delta_μ^μ$

but $\frac{\partial}{\partial x_μ} x_μ' = \alpha_{μλ} \gamma^{-1} = \alpha_{μλ}(L)$

$= \alpha_{μλ}(L) \frac{\partial}{\partial x_λ}$

So transforms same as $x_μ$

$\left( \frac{\partial}{\partial x_μ} \right)^2 = \nabla^2 - \frac{i}{c^2} \frac{\partial^2}{\partial t^2}$ wave equation operator!

inner products

If $u_μ$ and $v_μ$ are 4-vectors, then

$u_μ v_μ$ is Lorentz invariant scalar
Electromagnetism

Clearly $\vec{E} + \vec{B}$ must transform into each other under Lorentz tranform.

In frame $K$, stationary line charge $\lambda$

in frame $K'$, moving with $\vec{v} \parallel$ to wire

Stationary line charge gives current $\Rightarrow \vec{B}$ circulating around wire as well as outward radial $\vec{E}$

Loerntz force

$$\vec{F} = q \vec{E} + q \frac{\vec{v} \times \vec{B}}{c}$$

What is the velocity $\vec{v}$ here? Velocity with respect to what metric frame? Clearly $\vec{E}$ at $\vec{B}$ must change from metric frame to another if this force law can make sense.

Charge density

Consider charge $\Delta q$ contained in a vol $\Delta V$.

$\Delta q$ is a Lorentz invariant scalar.

Consider the reference frame in which the charge is instantaneously at rest. In this frame
\[ \Delta Q = \hat{\rho} \Delta V \]

\( \hat{\rho} \) is charge density in the rest frame.
\( \Delta V \) is volume in the rest frame.

\( \hat{\rho} \) is Lorentz invariant by definition.

Now transform to another frame moving with \( \vec{v} \) with respect to rest frame.

\[ \Delta Q \text{ remains the same} \]

\[ \Delta V = \frac{\Delta V}{\gamma} \] volume contracts in direction \( \vec{v} \) to \( \vec{v} \).

\[ \hat{\rho} = \frac{\Delta Q}{\Delta V} \]

Current density \( \vec{j} \) \( \vec{j} = \hat{\rho} \vec{v} = \gamma \vec{v} \cdot \hat{\rho} = \gamma \hat{\rho} \vec{u} \).

Define 4-current \[ j_\mu = (\vec{j}, i e \phi) = \hat{\rho}(\vec{u}, i e \phi) \]

\[ = \hat{\rho} u_\mu \]

it is a 4-vector since \( u_\mu \) is a 4-vector and \( \hat{\rho} \) is Lorentz invariant scalar.

Charge conservation

\[ \nabla \cdot \vec{j} + \frac{\partial \phi}{\partial t} = \frac{\partial j_\mu}{\partial x^\mu} = 0 \]

\[ \text{change to} \]

\[ \frac{\partial \phi}{\partial t} = 0 \]
Equation for potentials in Lorentz gauge

\[
(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \vec{A} = -\frac{4\pi}{c} \vec{J}
\]

\[
(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \phi = -4\pi \rho
\]

\[
\frac{\partial^2}{\partial x^\mu} = (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \text{ is Lorentz invariant operator}
\]

4-potential

\[
A_\mu = (\vec{A}, \phi)
\]

\[
(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) A_\mu = -\frac{4\pi}{c} j_\mu = \frac{\partial^2 A_\mu}{\partial x^\mu \partial t^2}
\]

Lorentz gauge condition is

\[
\nabla \cdot \vec{A} + \frac{\partial \phi}{\partial t} = \frac{\partial A_\mu}{\partial x^\mu} = 0
\]

Electric and magnetic fields

\[
B_i = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k}
\]

\[
E_i = -\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} = \epsilon_j k \text{ cyclic permutation of } i, j, k
\]

Define field stress tensor

\[
F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} = -F_{\nu\mu}
\]

\[
F_{\mu\nu} = \begin{pmatrix}
0 & B_3 & -B_2 & -\epsilon E_1 \\
-B_3 & 0 & B_1 & -\epsilon E_2 \\
B_2 & -B_1 & 0 & -\epsilon E_3 \\
\epsilon E_1 & \epsilon E_2 & \epsilon E_3 & 0
\end{pmatrix}
\]

"Curl" of a 4-vector is a 4x4 anti-symmetric 2nd rank tensor.
Inhomogeneous Maxwell's equations can be written in the form

\[
\frac{\partial F_{\mu\nu}}{\partial x^\mu} = 4\pi j^\mu - \frac{\partial E^\mu}{\partial t} = \frac{4\pi}{c} j^\mu
\]

\[
\frac{\partial E^\mu}{\partial x^\nu} - \frac{\partial B^\nu}{\partial x^\mu} = \frac{4\pi}{c} j^\mu
\]

\[
\Rightarrow \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\nu} = \frac{4\pi}{c} j^\mu
\]

agrees with previous equation for \( A^\mu \).

Transformation law for 2nd rank tensor \( F_{\mu\nu} \)

\[
F'_{\mu\nu} = \frac{\partial x'}{\partial x^\mu} \frac{\partial x'}{\partial x^\nu} F_{\mu\nu}
\]

use \( A'_\mu = a_\mu \sigma A_\sigma \)

\[
\frac{\partial x'}{\partial x^\mu} = a_\mu \sigma \frac{\partial x'}{\partial x^\sigma}
\]

\[
A'_\mu = a_\mu \sigma A_\sigma
\]

\[
F'_{\mu\nu} = a_\mu \sigma a_\nu \lambda F_{\sigma\lambda}
\]

\[
F'_{\mu\nu} = a_\mu \sigma a_\nu \lambda F_{\sigma\lambda}
\]

\[
F'_{\mu\nu} = a_\mu \sigma a_\nu \lambda F_{\sigma\lambda}
\]

For \( n \)th rank tensor, at one knows \( E \) and \( B \)

\[
T_{\mu_1 \mu_2 \ldots \mu_n} = a_{\mu_1 \nu_1} a_{\mu_2 \nu_2} \ldots a_{\mu_n \nu_n} T_{\nu_1 \nu_2 \ldots \nu_n}
\]
\[ \delta F_{\mu} = a_{\mu} \sigma \, a_{\nu} \, a_{\lambda} \, \frac{\delta F_{\sigma}}{\delta x} \]

but \( a_{\alpha} = a_{\alpha}^{-1} \) since inverse = transpose
\[ a_{\nu} \, a_{\lambda} = a_{\alpha} \, a_{\beta} = \delta_{\alpha}^{\beta} \]

\[ \frac{\delta F_{\mu}}{\delta x} = a_{\mu} \, \frac{\delta F_{\sigma}}{\delta x} \]

transforms like 4-vector

To write the homogeneous Maxwell Equations

Construct 3rd rank co-variant tensor

\[ \left[ G_{\mu \nu \lambda} = \frac{\delta F_{\mu}}{\delta x} + \frac{\delta F_{\nu}}{\delta x} + \frac{\delta F_{\lambda}}{\delta x} \right] \]

transforms as \( G_{\mu \nu \lambda} = a_{\mu} \, a_{\nu} \, a_{\lambda} \, G_{\alpha \beta \gamma} \)

in principle \( G \) has \( 4^3 = 64 \) Components

But can show that \( G \) is antisymmetric in exchange of any two indices

\[ G_{\mu \nu \lambda} = \frac{\delta F_{\mu}}{\delta x} + \frac{\delta F_{\nu}}{\delta x} + \frac{\delta F_{\lambda}}{\delta x} \]

\[ = - \frac{\delta F_{\nu}}{\delta x} - \frac{\delta F_{\lambda}}{\delta x} \quad \text{as } F \text{ anti-symmetric} \]

\[ \therefore - G_{\nu \lambda \mu} = G_{\mu \nu \lambda} \]
Also, $G_{\mu\nu} = 0$ at any two indices are equal

$\Rightarrow$ only 4 independent components

$G_{121}, G_{131}, G_{231}, G_{123}$

all other components either vanish or are $\pm$ one of the above.

The 4 homogeneous Maxwell Equations:

$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0$

can be written as

$G_{\mu\nu\lambda} = 0$

To see, substitute in definition of $G$ the definition of $F$

$G_{\mu\nu\lambda} = \frac{\partial^2 A_{\nu}}{\partial x_\lambda \partial x_\mu} - \frac{\partial^2 A_{\mu}}{\partial x_\nu \partial x_\lambda} + \frac{\partial^2 A_{\lambda}}{\partial x_\nu \partial x_\mu} - \frac{\partial^2 A_{\mu}}{\partial x_\nu \partial x_\lambda}

all terms cancel in pairs

$= 0$

$G_{123} = 0 \Rightarrow \nabla \cdot \vec{B} = 0$

$G_{121} = -\frac{i}{c} \left[ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \right]_{x=0} = 0 \quad \text{component Faraday's law}$