Special Relativity

1) Speed of light is constant in all inertial frames of reference
2) Physical laws must look the same in all inertial frames of refence - there is no experiment that can determine the "absolute" velocity of any inertial frame

$\Rightarrow$ If a flash of light goes off at the origin of some coord system, the outgoing wavefronts look spherical in all inertial frames.

Equation of wavefront is $r^2 - c^2t^2 = 0$

$\Rightarrow$ (x, y, z, t) coords in one inertial frame K

(x', y', z', t') coords in another inertial frame K' that moves with velocity $\vec{v} = v\hat{x}$ with respect to K.

What is the transformation that relates coords in K' to coords in K

$y = y'$, $z = z'$ (origins of K and K' coincide when $t = t' = 0$)

$$\Rightarrow \quad c^2t^2 - x^2 = c^2t'^2 - x'^2$$

$$\Rightarrow \quad \frac{(ct + x)}{(ct' + x')} \frac{(ct - x)}{(ct' - x')} = 1$$

Expect transformation to be <u>linear</u> (otherwise particle moving at constant $\vec{v}$ in one frame might look accelerated in another frame)

$$\Rightarrow \quad ct' + x' = (ct + x)\, f$$

$$ct' - x' = (ct - x)\, f^{-1}$$

for some constant f. Write $f = e^{-y}$ $\quad$ y is <u>rapidity</u>

Solve for ct' and x' in terms of ct and x

$$ct' = ct\left(\frac{e^y + e^{-y}}{2}\right) - x\left(\frac{e^y - e^{-y}}{2}\right)$$

$$x' = -ct\left(\frac{e^y - e^{-y}}{2}\right) + x\left(\frac{e^y + e^{-y}}{2}\right)$$

$$ct' = ct\cosh y - x\sinh y$$
$$x' = -ct\sinh y + x\cosh y$$

<u>meaning of parameter y</u>

the origin of K (at $x=0$) has trajectory $x' = -vt'$ in K'

$$\Rightarrow \frac{x'}{t'} = -v$$

from transformation above, with $x=0$, we get

$$\frac{x'}{ct'} = \frac{-ct\sinh y}{ct\cosh y} = -\tanh y$$

So $\frac{v}{c} = \tanh y$

$$\Rightarrow \cosh y = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \equiv \gamma$$

$$\sinh y = \left(\frac{v}{c}\right)\gamma$$

<u>Lorentz Transformation</u>
$$\begin{cases} ct' = \gamma ct - \gamma\left(\frac{v}{c}\right)x \\ x' = -\gamma\left(\frac{v}{c}\right)ct + \gamma x \end{cases}$$

Inverse transform obtained by taking $v \to -v$ in above

$$\begin{cases} ct = \gamma ct' + \gamma\left(\frac{v}{c}\right) x' \\ x = \gamma\left(\frac{v}{c}\right) ct' + \gamma x' \end{cases}$$

<u>4 - vectors</u>

<u>4-position:</u> $x_\mu = (x_1, x_2, x_3, ict)$ $\quad x_4 \equiv ict$

$$x_\mu x_\mu \equiv \sum_{\mu=1}^{4} x_\mu^2 = r^2 - c^2 t^2$$

Lorentz invariant scalar – has same value in all inertial frames

Lorentz transf is

$$\left.\begin{aligned} x_1' &= \gamma\left(x_1 + i\left(\tfrac{v}{c}\right) x_4\right) \\ x_2' &= x_2 \\ x_3' &= x_3 \\ x_4' &= \gamma\left(x_4 - i\left(\tfrac{v}{c}\right) x_1\right) \end{aligned}\right\}$$

linear transf, can be represented by a matrix

or $\quad x_\mu' = a_{\mu\nu}(L)\, x_\nu$

↑ matrix of Lorentz transformation L

$$a(L) = \begin{pmatrix} \gamma & 0 & 0 & i\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix}$$

inverse: $\quad x_\mu = a_{\mu\nu}(L^{-1})\, x_\nu'$

$a_{\mu\nu}(L^{-1})$ is given by taking $v \to -v$ in $a_{\mu\nu}(L)$

we see $\quad a_{\mu\nu}(L^{-1}) = a_{\nu\mu}(L)$

inverse = transpose

More generally

Since $x_\mu'^2$ is Lorentz invariant scalar,

$$x_\mu'^2 = a_{\mu\nu}(L)\, a_{\mu\lambda}(L)\, x_\nu x_\lambda = x_\lambda^2$$

$$\Rightarrow a_{\mu\nu}(L)\, a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow a^t_{\nu\mu}(L)\, a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow a^t_{\mu\nu} = a^{-1}_{\mu\nu}(L)$$ transpose = inverse

$a_{\mu\nu}$ is 4x4 orthogonal matrix

If L_1 is a Lorentz transf from K to K′

L_2 is a Lorentz transf from K′ to K″

Then the Lorentz transf from K to K″ is given by the matrix

$$a(L_2 L_1) = a(L_2)\, a(L_1)$$

if $L_1 \equiv L$ and $L_2 = L^{-1}$ so $L_2 L_1 = I$ identity

$$\Rightarrow a^{-1}(L) = a(L^{-1})$$

$$dx_\mu = (dx_1, dx_2, dx_3, icdt)$$

$$-(dx_\mu)^2 \equiv c^2 ds^2 = c^2 dt^2 - dr^2$$ Lorentz invariant scalar

$$ds^2 = dt^2\left[1 - \frac{1}{c^2}\left(\frac{dx_1}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dx_2}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dx_3}{dt}\right)^2\right]$$

$$ds^2 = \frac{dt^2}{\gamma^2}$$

$$\boxed{ds = \frac{dt}{\gamma}}$$ proper time interval

A 4-vector is any 4 numbers that transform under a Lorentz transformation the same way as does x_μ

<u>4-velocity</u> $\quad u_\mu \equiv \dfrac{dx_\mu}{ds} \equiv \dot{x}_\mu$

$$= \gamma \frac{dx_\mu}{dt}$$

space components $\vec{u} = \gamma \vec{v}$

$$u_4 = ic\gamma$$

$$u_\mu u_\mu = \gamma^2 v^2 - c^2\gamma^2 = \gamma^2(v^2 - c^2)$$
$$= \frac{v^2 - c^2}{1 - \frac{v^2}{c^2}} = -c^2$$

<u>4-acceleration</u> $\quad \alpha_\mu \equiv \dfrac{du_\mu}{ds} = \gamma \dfrac{du_\mu}{dt}$

<u>4-gradient</u> $\quad \dfrac{\partial}{\partial x_\mu} \equiv \left(\vec{\nabla}, \dfrac{-i}{c}\dfrac{\partial}{\partial t}\right)$

proof $\dfrac{\partial}{\partial x_\mu}$ is a 4-vector

$$\frac{\partial}{\partial x_\mu'} = \frac{\partial x_\lambda}{\partial x_\mu'} \frac{\partial}{\partial x_\lambda} \qquad \text{but } \frac{\partial x_\lambda}{\partial x_\mu'} = a_{\lambda\mu}(L^{-1}) = a_{\mu\lambda}(L)$$

$$= a_{\mu\lambda}(L) \frac{\partial}{\partial x_\lambda}$$

so transforms same as x_μ

$$\left(\frac{\partial}{\partial x_\mu}\right)^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \qquad \text{wave equation operator!}$$

<u>inner products</u>

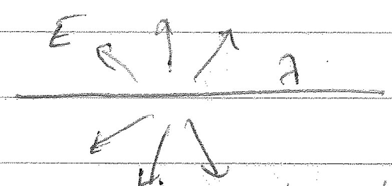
If U_μ and V_μ are 4-vectors, then $U_\mu V_\mu$ is Lorentz invarient scalar

Electromagnetism

Clearly $\vec{E}$ + $\vec{B}$ must transform into each other under Lorentz transf.

in inertial frame K

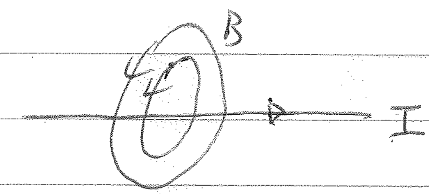
stationary line charge λ



cylindrical outward electric field

no B-field

in frame K' moving with $\vec{v}$ ∥ to wire



moving line charge gives current

⇒ B circulating around wire as well as outward radial E

Lorentz force

$$\vec{F} = q\vec{E} + q\frac{\vec{v}}{c}\times\vec{B}$$

What is the velocity $\vec{v}$ here? velocity with respect to what inertial frame? Clearly $\vec{E}$ and $\vec{B}$ must change from one inertial frame to another if this force law can make sense.

Charge density

Consider charge ΔQ contained in a vol ΔV.

ΔQ is a Lorentz invariant scalar.

Consider the reference frame in which the charge is instantaneously at rest. In this frame

$$\Delta Q = \mathring{\rho}\, \Delta \mathring{V}$$ $\mathring{\rho}$ is charge density in the rest frame

$\Delta\mathring{V}$ is volume in the rest frame

$\mathring{\rho}$ is Lorentz invariant by definition

Now transform to another frame moving with $\vec{v}$ with respect to rest frame

ΔQ remains the same

$$\Delta V = \frac{\Delta \mathring{V}}{\gamma}$$ volume contracts in direction $\parallel$ to $\vec{v}$

$$\rho = \frac{\Delta Q}{\Delta V} = \frac{\Delta Q}{\Delta \mathring{V}}\gamma = \mathring{\rho}\gamma$$

Current density is $\vec{j} = \rho \vec{v} = \gamma \vec{v} \cdot \frac{\rho}{\gamma} = \mathring{\rho}\vec{u}$

Define 4-current $\boxed{j_\mu = (\vec{j}, ic\rho)} = \mathring{\rho}(\vec{u}, ic\gamma)$

$= \mathring{\rho} u_\mu$

It is 4-vector since u_μ is 4-vector and $\mathring{\rho}$ is Lorentz invariant scalar.

charge conservation

$$\vec{\nabla}\cdot\vec{j} + \frac{\partial \rho}{\partial t} = \boxed{\frac{\partial j_\mu}{\partial x_\mu} = 0}$$

Equation for potentials in Lorentz gauge

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\frac{4\pi}{c}\vec{j}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -4\pi\rho$$

$\frac{\partial^2}{\partial x_\mu^2} = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)$ is Lorentz invariant operator

4-potential $\boxed{A_\mu = (\vec{A},\ i\phi)}$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A_\mu = \boxed{-\frac{4\pi}{c} j_\mu = \frac{\partial^2 A_\mu}{\partial x_{\cancel{\lambda}}^2}}$$

Lorentz gauge condition is

$$\vec{\nabla}\cdot\vec{A} + \frac{\partial \phi}{c\,\partial t} = \frac{\partial A_\mu}{\partial x_\mu} = 0$$

Electric and magnetic fields

$$B_i = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \qquad i, j, k \text{ cyclic permutation of } 1, 2, 3$$

$$E_i = -\frac{\partial \phi}{\partial x_i} - \frac{\partial A_i}{c\,\partial t} = i\left(\frac{\partial A_4}{\partial x_i} - \frac{\partial A_i}{\partial x_4}\right)$$

Define field stress tensor

$$\boxed{F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = -F_{\nu\mu}}$$

$$= \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix}$$

"curl" of a 4-vector is a 4x4 anti symmetric 2nd rank tensor

Inhomogeneous Maxwell's equations can be written in the form

$$\boxed{\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi}{c} j_\mu} \Rightarrow \left[\begin{array}{l} \vec{\nabla}\cdot\vec{E} = 4\pi\rho \\ \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j} \end{array}\right]$$

$$= \frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right) = \frac{\partial}{\partial x_\mu}\underbrace{\left(\frac{\partial A_\nu}{\partial x_\nu}\right)}_{=0} - \frac{\partial^2 A_\mu}{\partial x_\nu^2}$$

$$\Rightarrow \quad -\frac{\partial^2 A_\mu}{\partial x_\nu^2} = \frac{4\pi}{c} j_\mu$$ agrees with previous equation for A_μ

transformation law for 2nd rank tensor $F_{\mu\nu}$

$$F'_{\mu\nu} = \frac{\partial A'_\nu}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\nu}$$

use $A'_\mu = a_{\mu\sigma} A_\sigma$

$$\frac{\partial}{\partial x'_\mu} = a_{\mu\lambda}\frac{\partial}{\partial x_\lambda}$$

$$= a_{\nu\lambda} a_{\mu\sigma} \frac{\partial A_\lambda}{\partial x_\sigma}$$

$$- a_{\mu\sigma} a_{\nu\lambda} \frac{\partial A_\sigma}{\partial x_\lambda}$$

$$F'_{\mu\nu} = a_{\mu\sigma} a_{\nu\lambda} F_{\sigma\lambda}$$

lets one find $\vec{E}'$ and $\vec{B}'$ if one knows $\vec{E}$ and $\vec{B}$

For n^{th} rank tensor

$$T'_{\mu_1\mu_2\cdots\mu_n} = a_{\mu_1\nu_1} a_{\mu_2\nu_2} \cdots a_{\mu_n\nu_n} T_{\nu_1\nu_2\cdots\nu_n}$$

$\frac{\partial F_{\mu\nu}}{\partial x_\nu}$ is a 4-vector: proof

$$\frac{\partial F'_{\mu\nu}}{\partial x'_\nu} = a_{\mu\sigma}\, a_{\nu\lambda}\, a_{\nu\gamma} \frac{\partial F_{\sigma\lambda}}{\partial x_\gamma}$$

but $a_{\nu\lambda} = a^{-1}_{\lambda\nu}$ since inverse = transpose

$$a_{\nu\lambda}\, a_{\nu\gamma} = a^{-1}_{\lambda\nu}\, a_{\nu\gamma} = \delta_{\lambda\gamma}$$

$$\frac{\partial F'_{\mu\nu}}{\partial x'_\nu} = a_{\mu\sigma} \frac{\partial F_{\sigma\lambda}}{\partial x_\gamma} \delta_{\lambda\gamma} = a_{\mu\sigma} \frac{\partial F_{\sigma\lambda}}{\partial x_\lambda}$$ transforms like 4-vector

To write the <u>homogeneous Maxwell Equations</u>

Construct 3rd rank co-variant tensor

$$\boxed{G_{\mu\nu\lambda} = \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu}}$$

transforms as $G'_{\mu\nu\lambda} = a_{\mu\alpha}\, a_{\nu\beta}\, a_{\lambda\gamma}\, G_{\alpha\beta\gamma}$

in principle G has $4^3 = 64$ components

But can show that G is antisymmetric in exchange of any two indices

$$G_{\nu\mu\lambda} = \frac{\partial F_{\nu\mu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\nu}}{\partial x_\mu} + \frac{\partial F_{\mu\lambda}}{\partial x_\nu}$$

$$= -\frac{\partial F_{\mu\nu}}{\partial x_\lambda} - \frac{\partial F_{\nu\lambda}}{\partial x_\mu} - \frac{\partial F_{\lambda\mu}}{\partial x_\nu}$$ as F antisymmetric

$$= -G_{\mu\nu\lambda}$$

also $G_{\mu\nu\lambda} = 0$ if any two indices are equal

$\Rightarrow$ only 4 independent components

G_{412}, G_{413}, G_{423}, G_{123}

all other components either vanish or are $\pm$ one of the above.

the 4 homogeneous Maxwell Equations:

$$\vec{\nabla}\cdot\vec{B} = 0 \quad , \quad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

can be written as

$$\boxed{G_{\mu\nu\lambda} = 0}$$

to see, substitute in definition of G the definition of F

$$G_{\mu\nu\lambda} = \frac{\partial^2 A_\nu}{\partial x_\lambda \partial x_\mu} - \frac{\partial^2 A_\mu}{\partial x_\lambda \partial x_\nu} + \frac{\partial^2 A_\mu}{\partial x_\nu \partial x_\lambda} - \frac{\partial^2 A_\lambda}{\partial x_\nu \partial x_\mu} + \frac{\partial^2 A_\lambda}{\partial x_\mu \partial x_\nu} - \frac{\partial^2 A_\nu}{\partial x_\mu \partial x_\lambda}$$

all terms cancel in pairs

$= 0$

$G_{123} = 0 \Rightarrow \vec{\nabla}\cdot\vec{B} = 0$

$G_{412} = -i\left[\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{c\,\partial t}\right]_z = 0$ z component Faraday's law