Solutions PHY 415 Final Exam

1) a) Take coordinates with the atom at the origin and the charge q at position $\vec{r}_q = r\hat{z}$ on the z-axis. The electric field from q at an arbitrary position $\vec{r}$ is then

$$\vec{E}_q(\vec{r}) = q\frac{(\vec{r}-\vec{r}_q)}{|\vec{r}-\vec{r}_q|^3} = \frac{q(\vec{r}-r\hat{z})}{|\vec{r}-r\hat{z}|^3}$$

So the field at the atom is $\vec{E}_q(0) = -\frac{q\hat{z}}{r^2}$ and this field polarizes the atom which then develops a dipole moment $\vec{p} = \alpha\vec{E}_q(0) = -\frac{\alpha q\hat{z}}{r^2}$

The atomic dipole moment $\vec{p}$ produces an electric field

$$\vec{E}_p(\vec{r}) = \frac{3(\vec{p}\cdot\hat{r})\hat{r}-\vec{p}}{r^3} = \frac{-\alpha q}{r^2}\,\frac{3(\hat{z}\cdot\hat{r})\hat{r}-\hat{z}}{r^5}$$

The force on the charge is due to $\vec{E}_p$ and is

$$\vec{F} = q\vec{E}_p(\vec{r}_q) = -\frac{\alpha q^2}{r^5}\left[3(\hat{z}\cdot\hat{z})\hat{z}-\hat{z}\right] \quad \text{since } \hat{r}_q = \hat{z}$$

$$\boxed{\vec{F} = -\frac{2\alpha q^2}{r^5}\hat{z}}$$

So force $\sim \frac{1}{r^5}$ is <u>attractive</u> independent of the sign of q.

An alternative method would be to compute the force on the dipole due to the $\vec{E}$ field of the charge q

$$\vec{F}_{dip} = (\vec{p} \cdot \vec{\nabla})\vec{E}_q$$

let the charge q be at position $\vec{r}_q = s\hat{z}$ and the dipole is at the origin

then $\vec{p} = -\dfrac{\alpha q}{s^2}\hat{z}$ so

$$\vec{F}_{dip} = -\frac{\alpha q}{s^2}(\hat{z}\cdot\vec{\nabla})\vec{E}_q = -\frac{\alpha q}{s^2}\left(\frac{\partial}{\partial z}\vec{E}_q\right)_{\vec{r}=0}$$

Now $\vec{E}_q(\vec{r}) = q\dfrac{\vec{r}-s\hat{z}}{|\vec{r}-s\hat{z}|^3} = q\dfrac{x\hat{x}+y\hat{y}+(z-s)\hat{z}}{[x^2+y^2+(z-s)^2]^{3/2}}$

So $$\frac{\partial\vec{E}_q}{\partial z} = q\,\frac{[x^2+y^2+(z-s)^2]^{3/2}\,\hat{z} - (x\hat{x}+y\hat{y}+(z-s)\hat{z})\frac{3}{2}[\quad]^{1/2}\,2(z-s)}{[x^2+y^2+(z-s)^2]^3}$$

evaluate at $\vec{r} = (x,y,z) = 0$ to get derivative at the dipole

$$\left(\frac{\partial\vec{E}_q}{\partial z}\right)_{\vec{r}=0} = q\,\frac{[s^2]^{3/2}\hat{z} - (-s\hat{z})\,3\,[s^2]^{1/2}(-s)}{[s^2]^3}$$

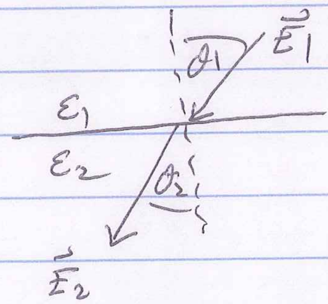
$$= q\hat{z}\,\frac{s^3-3s^3}{s^6} = -\frac{2q}{s^3}\hat{z}$$

So

$$\vec{F}_{dip} = \left(-\frac{\alpha q}{s^2}\right)\left(-\frac{2q}{s^3}\right)\hat{z} = \frac{2\alpha q^2}{s^5}\hat{z}$$

force on dipole is in $+\hat{z}$ so pulling it towards q, ie force is attractive. From Newton's 3rd law, force on q is

$$\vec{F}_q = -\vec{F}_{dip} = -\frac{2\alpha q^2}{s^5}\hat{z}$$ as found before

b) 

① At the interface the tangential component of $\vec{E}$ is continuous

② Since there is no free charge at the interface, the normal component of $\vec{D}$ is continuous

① $\Rightarrow$ $E_1 \sin\theta_1 = E_2 \sin\theta_2$

② $\Rightarrow$ $D_1 \cos\theta_1 = D_2 \cos\theta_2$ use $\vec{D}_1 = \varepsilon_1 \vec{E}_1$, $\vec{D}_2 = \varepsilon_2 \vec{E}_2$

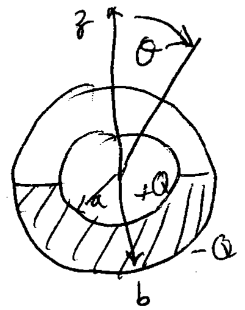
$\Rightarrow$ $\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2$

divide to get $\varepsilon_1 \cot\theta_1 = \varepsilon_2 \cot\theta_2$

$$\Rightarrow \frac{\cot\theta_1}{\cot\theta_2} = \boxed{\frac{\tan\theta_2}{\tan\theta_1} = \frac{\varepsilon_2}{\varepsilon_1}}$$

$$\boxed{\theta_2 = \arctan\left(\frac{\varepsilon_2}{\varepsilon_1}\tan\theta_1\right)}$$

2)



Assume solution is a radial electric field

$\vec{E}(\vec{r}) = E(r)\,\hat{r}$ same for all θ

$\Rightarrow$ total charge density σ_{tot} is uniform on the inner and outer spheres

and $E(r) = \dfrac{4\pi a^2 \sigma_{tot}}{r^2}$ $\quad a < r < b$

at $r = a$ $\quad \sigma_{tot} = \begin{cases} \sigma_{free} & 0 < \theta < \frac{\pi}{2} \\ \sigma'_{free} + \sigma_b & \frac{\pi}{2} < \theta < \pi \end{cases}$

where we know $2\pi a^2 (\sigma_{free} + \sigma'_{free}) = Q$ total free charge

and for a linear dielectric, free charge is screened by ϵ, so

$$\sigma_{tot} = \frac{\sigma'_{free}}{\epsilon}$$

so

$$2\pi a^2 (\sigma_{free} + \sigma'_{free}) = 2\pi a^2 (\sigma_{tot} + \epsilon\,\sigma_{tot}) = 2\pi a^2 (1+\epsilon)\,\sigma_{tot}$$
$$= Q$$

$$\Rightarrow \boxed{\sigma_{tot} = \frac{Q}{2\pi a^2 (1+\epsilon)}}$$

a) $\boxed{\vec{E} = \frac{2Q}{(1+\epsilon)r^2}\,\hat{r}}$ $\quad a<r<b$

b) $\boxed{\sigma_{tot} = \frac{Q}{2\pi a^2(1+\epsilon)}}$ $\quad r=a$

c) $\sigma_b = \sigma_{tot} - \sigma'_{free} = \sigma_{tot} - \epsilon\,\sigma_{tot} = (1-\epsilon)\,\sigma_{tot}$

$$\boxed{\sigma_b = \frac{(1-\epsilon)\,Q}{2\pi a^2(1+\epsilon)}}$$

above gives a self consistent solution therefore it is the unique solution!

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to prove that $\vec{E}(\vec{r}) = E(r)\hat{r} = \frac{Q_{tot}}{r^2}\hat{r}$

note that the total charge in the region $a<r<b$ vanishes since there is no free charge there and $\rho_{tot} = \frac{\rho_{free}}{\epsilon}$

$\Rightarrow \nabla^2\phi = 0$ in this region where ϕ is the usual electrostatic potential, $\vec{E} = -\vec{\nabla}\phi$

Azimuthal symmetry $\Rightarrow \phi(r,\theta) = \sum_\ell \left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right]P_\ell(\cos$

At $r=a$, ϕ is a constant, since surface of sphere is equipotential

$$\Rightarrow \phi(a,\theta) = \sum_\ell \left[A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}}\right]P_\ell(\cos\theta)$$

must be independent of θ.

$\Rightarrow$ coefficients of P_ℓ must vanish for $\ell \neq 0$

$$\Rightarrow \quad A_\ell = \frac{-B_\ell}{a^{2\ell+1}}$$

but same arguement holds at $r = b$

$$\Rightarrow \quad A_\ell = -\frac{B_\ell}{b^{2\ell+1}}$$

only way to solve both conditions is $A_\ell = B_\ell = 0$ for $\ell \neq 0$

$$\Rightarrow \quad \phi(r,\theta) = \left[A_0 + \frac{B_0}{r}\right] \qquad \text{since } P_0(\cos\theta) = 1$$

$$\Rightarrow \quad \vec{E} = -\vec{\nabla}\phi = \frac{B_0}{r^2}\,\hat{r}$$

3) a) transparent region: $\varepsilon = \varepsilon_1 + i\varepsilon_2$ with $\varepsilon_2 \ll \varepsilon_1$

$k = k_1 + i k_2$ with $k_2 \ll k_1$

resonant absorption: $\varepsilon_2 \gg \varepsilon_1$

$k_1 \approx k_2$

total reflection $\varepsilon_2 \ll |\varepsilon_1|$ and $\varepsilon_1 < 0$

$k_2 \gg k_1$

b) For a conductor the effective dielectric function is

$$\varepsilon(\omega) = \varepsilon_b(\omega) + \frac{4\pi i \sigma(\omega)}{\omega}$$

from free conduction electrons

from bound electrons - like ε of a dielectric

One key difference between waves in a dielectric and in a conductor is that a dielectric is transparent at low frequencies below the resonant frequency $\omega \ll \omega_0$ while a conductor is strongly absorbing at low frequencies because $\sigma(0)$ is real so the contribution to ε from the conduction electrons is large and imaginary

c) $\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H}$ wave propagates along $\hat{z}$

$\Rightarrow \vec{k} = k\hat{z} = (k_1 + i k_2)\hat{z}$

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} = \vec{E}_0 e^{-k_2 z} e^{i(k z - \omega t)}$$

with $\vec{E}_0 \perp \hat{z}$

$$\vec{H} = \frac{c|k|}{\omega\mu} \hat{z} \times \vec{E}_0 e^{-k_2 z} e^{i(kz - \omega t + \delta)}$$

where $|k| = \sqrt{k_1^2 + k_2^2}$ and $\tan\delta = k_2/k_1$

To compute $\vec{S}$ we need to first take the real parts of the above complex expressions for $\vec{E}$ and $\vec{H}$

$$\vec{E} = \vec{E}_0 e^{-k_2 z} \cos(kz - \omega t)$$

$$\vec{H} = \frac{c|k|}{\omega\mu} \hat{z} \times \vec{E}_0 e^{-k_2 z} \cos(kz - \omega t + \delta)$$

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H} = \hat{z} \frac{c^2 |k| E_0^2 e^{-2k_2 z}}{4\pi\omega\mu} \cos(kz - \omega t) \cos(kz - \omega t + \delta)$$

Now in the region of total reflection $k_2 \gg k_1$

$\Rightarrow \tan\delta = \frac{k_2}{k_1} \gg 1 \quad \Rightarrow \delta \approx \frac{\pi}{2}$

$$\cos\left(\Phi + \frac{\pi}{2}\right) = \cos\Phi \cos\frac{\pi}{2} - \sin\Phi \sin\frac{\pi}{2} = -\sin\Phi$$

So

$$\vec{S} = -\hat{z}\,\frac{c^2|k|E_0^2}{4\pi\omega\mu}\,e^{-2k_2 z}\cos(kz-\omega t)\sin(kz-\omega t)$$

taking the time average $\langle \vec{S} \rangle \sim \langle \cos(\Phi)\sin(\Phi) \rangle$

$\Phi = kz - \omega t$

$$\langle \cos\Phi \sin\Phi \rangle = \frac{1}{2\pi}\int_0^{2\pi} d\Phi\, \cos\Phi \sin\Phi$$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\Phi\, \sin 2\Phi = 0$$

So $\langle \vec{S} \rangle = 0$

This is consistent with what we expect for a region of total reflection — no energy is transported into the material!

4) $I(\phi, t) = \mathrm{Re}\left\{ I_0 \cos(n\phi)\, e^{-i\omega t} \right\}$

$$= I_0 \cos(n\phi) \cos \omega t$$

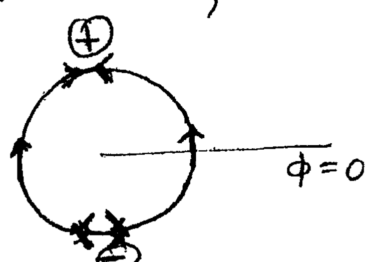
easy way to see the answer

a) when $n = 0$, $I = I_0 \cos \omega t$ const oscilating current

no charges are accumulated anywhere since
I is indep of ϕ

$\Rightarrow$ magnetic dipole but no electric dipole

b) when $n = 1$, $I = I_0 \cos(\phi) \cos \omega t$



current pattern looks like

charges build up at top and bottom and oscillate in time

$\Rightarrow$ electric dipole

but $\vec{m} = \frac{1}{2c} \int dl\, \vec{r} \times \vec{I}(\phi)$

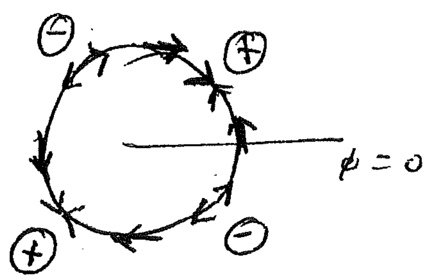
$$= \frac{1}{2c} R \int R\, d\phi\, I_0 \cos\phi\, \hat{r} \times \hat{\phi}$$

$$= \frac{I_0}{2c} R^2 \hat{z} \int_0^{2\pi} d\phi \cos\phi = 0$$

no magnetic moment

c) when $n=2$, $I = I_0 \cos 2\phi \cos \omega t$

pattern of current looks like



charges build up as shown
$\Rightarrow$ oscillating
electric quadrupole

Still magnetic dipole is

$$\vec{m} = \frac{I_0}{2c} R^2 \hat{z} \int_0^{2\pi} d\phi \cos 2\phi = 0$$

no electric or magnetic dipoles

radiation is emitted with freq ω.

The above is all you needed to say for the exam. But we can do the explicit calculations:

More formally

as above magnetic dipole moment is

$$\vec{m} = \frac{1}{2c} \int d^3r \, \vec{r} \times \vec{J} = \frac{1}{2c} \oint dl \, \vec{r} \times I(\phi)$$

$$= \frac{I_0}{2c} R^2 \hat{z} \oint_0^{2\pi} d\phi \cos(n\phi) = 0$$

except for $n=0$

for $n=0$, $\boxed{\vec{m} = \frac{\pi R^2 I_0}{c} \hat{z}}$ = current × area

for $\boxed{n \neq 0, \ \vec{m} = 0}$

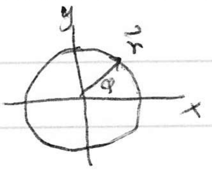
To compute the <u>electric dipole moment</u>

$$\vec{p}_\omega = \int d^3r \, \vec{r} \, \rho_\omega(\vec{r})$$

For an oscillating current $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \Rightarrow -i\omega \rho_\omega + \vec{\nabla}\cdot\vec{j}_\omega(\vec{r}) = 0$

$$\rho_\omega = \frac{\vec{\nabla}\cdot\vec{j}_\omega}{i\omega}$$

For a current in a circular loop in the xy plane the above becomes for the line charge λ_ω



$$\lambda_\omega = \frac{1}{i\omega}\frac{1}{R}\frac{\partial I_\omega}{\partial \varphi}$$ where we used $\vec{\nabla}\cdot \rightarrow \frac{1}{R}\frac{\partial}{\partial\varphi}$ in cylindrical coords for the loop

$$\vec{p}_\omega = \oint dl \, \vec{r} \, \lambda_\omega = \int_0^{2\pi} d\varphi \, R \left[R\cos\varphi\,\hat{x} + R\sin\varphi\,\hat{y}\right] \lambda_\omega(\varphi)$$

$$= \frac{1}{i\omega R}\int_0^{2\pi} d\varphi \, R \left[R\cos\varphi\,\hat{x} + R\sin\varphi\,\hat{y}\right]\frac{\partial I_\omega}{\partial\varphi}$$

$$= \frac{R I_0}{i\omega}\int_0^{2\pi} d\varphi \left[\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}\right](-n\sin n\varphi)$$

$$\vec{p}_\omega = -\frac{nRI_0}{i\omega}\int_0^{2\pi} d\varphi \left[\sin n\varphi\cos\varphi\,\hat{x} + \sin n\varphi\sin\varphi\,\hat{y}\right]$$

$\Rightarrow$ <u>For $n=0$</u>, $\boxed{\vec{p}_\omega = 0}$

<u>For $n=1$</u>, $\vec{p}_\omega = -\frac{RI_0}{i\omega}\int_0^{2\pi} d\varphi\left[\sin\varphi\cos\varphi\,\hat{x} + \sin^2\varphi\,\hat{y}\right]$

$$\boxed{\vec{p}_\omega = -\frac{RI_0}{i\omega}\pi\hat{y}}$$

since $\int_0^{2\pi} d\varphi \sin\varphi\cos\varphi = 0$

$\int_0^{2\pi} d\varphi \sin^2\varphi = \pi$

For $n=2$, $\vec{p}_\omega = \frac{-2RI_0}{i\omega} \int_0^{2\pi} d\varphi \left[\sin 2\varphi \cos\varphi \, \hat{x} + \sin 2\varphi \sin\varphi \, \hat{y} \right]$

use $\sin 2\varphi = 2 \sin\varphi \cos\varphi$

$$\vec{p}_\omega = \frac{-2RI_0}{i\omega} \int_0^{2\pi} d\varphi \left[2 \sin\varphi \cos^2\varphi \, \hat{x} + 2 \sin^2\varphi \cos\varphi \, \hat{y} \right]$$

$$= \frac{-2RI_0}{i\omega} \, 2 \left[-\frac{\cos^3\varphi}{3} \, \hat{x} + \frac{\sin^3\varphi}{3} \, \hat{y} \right]_0^{2\pi} = 0$$

$$\boxed{\vec{p}_\omega = 0}$$

We can directly compute the amplitude of the electric quadrupole oscillation.

$$\overleftrightarrow{Q}_{ij} = \int d^3r' \left(3\vec{r}_i{}'\vec{r}_j{}' - r'^2\delta_{ij}\right)\rho(\vec{r}')$$

$$\vec{r}\,' = R(x, y, 0) = R(\cos\varphi, \sin\varphi, 0)$$

as in calculation of $\vec{P}\omega$,

use $\rho(\vec{r}')\,d^3r' = \dfrac{I_0[-n\sin(n\varphi)]}{i\omega R}\,dl$ $\qquad dl = R\,d\varphi$

$$Q_{xx} = \int_0^{2\pi} R\,d\varphi\left[3R^2\cos^2\varphi - R^2\right]\frac{I_0(-n\sin n\varphi)}{i\omega R}$$

$$= \frac{-I_0 n R^2}{i\omega}\int_0^{2\pi} d\varphi\left[3\cos^2\varphi\sin n\varphi - \sin n\varphi\right]$$

this term always integrates to zero for any n

$$= -\frac{I_0 n R^2}{i\omega}\,3\int_0^{2\pi} d\varphi\,\cos^2\varphi\sin n\varphi$$

$$Q_{yy} = \frac{-I_0 n R^2}{i\omega}\,3\int_0^{2\pi} d\varphi\,\sin^2\varphi\sin n\varphi$$

$$Q_{xy} = Q_{yx} = \frac{-I_0 n R^2}{i\omega}\,3\int d\varphi\,\sin\varphi\cos\varphi\sin n\varphi$$

$Q_{xz} = Q_{yz} = Q_{zx} = Q_{zy} = Q_{zz} = 0$ because $z' = 0$ and the contrib from $r'^2\delta_{ij}$ always vanishes

One can show that for $n = 0, 1$, then $\overleftrightarrow{Q} = 0$

For $n=2$

$$Q_{xx} \propto \int_0^{2\pi} d\varphi \cos^2\varphi \sin 2\varphi \qquad \text{use } \cos^2\varphi = \frac{1+\cos 2\varphi}{2}$$

$$= \frac{1}{2}\int_0^{2\pi} d\varphi \,(1+\cos 2\varphi)\sin 2\varphi$$

$$= \underbrace{\frac{1}{2}\int_0^{2\pi} d\varphi \sin 2\varphi}_{=0} + \frac{1}{2}\int_0^{2\pi} d\varphi \cos 2\varphi \sin 2\varphi$$

$$\frac{1}{4}\int_0^{2\pi} d\varphi \sin 4\varphi = 0$$

$$Q_{yy} \propto \int_0^{2\pi} d\varphi \sin^2\varphi \sin 2\varphi \qquad \text{use } \sin^2\varphi = \frac{1-\cos 2\varphi}{2}$$

$$= \frac{1}{2}\int_0^{2\pi} d\varphi \,(1-\cos 2\varphi)\sin 2\varphi = 0$$

$$Q_{xy} \propto \int_0^{2\pi} d\varphi \sin\varphi\cos\varphi \sin 2\varphi = \frac{1}{2}\int_0^{2\pi} d\varphi \sin 2\varphi \sin 2\varphi$$

$$= \frac{1}{2}\int_0^{2\pi} d\varphi \sin^2 2\varphi = \frac{1}{2}\cdot\frac{1}{2}\cdot 2\pi = \frac{\pi}{2}$$

$$Q_{xy} = -\frac{I_0 2R^2}{i\omega}\, 3\,\frac{\pi}{2} = -\frac{3\pi I_0 R^2}{i\omega} = \frac{i\,3\pi I_0 R^2}{\omega}$$

$$\overleftrightarrow{Q} = \frac{i\,3\pi I_0 R^2}{\omega}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$$

so for $n=2$ there is electric quadrapole radiation