PHY 415 **Final Exam** **Fall 2019**

• Please write and sign on the inside cover of your blue book the academic honesty pledge, "I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own."

• For each question, please put a box around your final answer, and cross out any work you do not wish me to look at. If you include sentences explaining your steps you are more likely to receive partial credit if you make a mistake.

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1) [30 points total]

Consider a dielectric sphere of radius R with dielectric constant ϵ. At the center of the sphere is a point charge q. The sphere is placed in a uniform external electric field $\mathbf{E}_0 = E_0\hat{z}$.

a) [20 points] What is the electrostatic potential inside and outside the sphere?

b) [10 points] What is the total surface charge density induced on the surface of the sphere?

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2) [35 points total]

For homework, you considered the effect of a static uniform magnetic field on the propagation of electromagnetic waves in a dielectric (the Faraday effect). In this problem, you will consider the effect of a static uniform magnetic field on the propagation of electromagnetic waves in a conductor. Assume we have a conductor with n conduction electrons per unit volume, and the effect of the bound electrons can be ignored (i.e. $\epsilon_b = 1$). Also assume $\mu = 1$. Suppose there is a constant and uniform magnetic field along the z axis, $\mathbf{B}_0 = B_0\hat{z}$.

a) [15 points] Consider now an oscillating electric field $\mathrm{Re}[\mathbf{E}_\omega e^{-i\omega t}]$ with $\mathbf{E}_\omega$ perpendicular to $\mathbf{B}_0$. For a circularly polarized electric field, i.e. $\mathbf{E}_\omega = E_0(\hat{x} \pm i\hat{y})$, show that the current density of the conduction electrons is $\mathrm{Re}[\mathbf{j}_\omega e^{-i\omega t}]$ where $\mathbf{j}_\omega = j_0(\hat{x} \pm i\hat{y})$, and

$$j_0 = \frac{(ne^2\tau/m)}{1 - i(\omega \mp \omega_c)\tau} E_0 \tag{1}$$

where $-e$ and m are the charge and mass of the conduction electrons, τ is the relaxation time of the electrons, and $\omega_c \equiv eB_0/mc$ is the cyclotron frequency. (*Hint*: treat the electrons as classical particles moving in the Lorentz force of $\mathbf{E}$ and $\mathbf{B}_0$ with a damping force $-m\mathbf{v}/\tau$.)

b) [10 points] Consider now a circularly polarized electromagnetic wave propagating in the z direction through the conductor, with electric field $\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}[E_0(\hat{x} \pm i\hat{y})e^{i(kz-\omega t)}]$. Show that k and ω are related by the dispersion relation $k^2 = (\omega/c)^2\epsilon(\omega)$, with

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega}\left(\frac{1}{\omega \mp \omega_c + i/\tau}\right) \ , \tag{2}$$

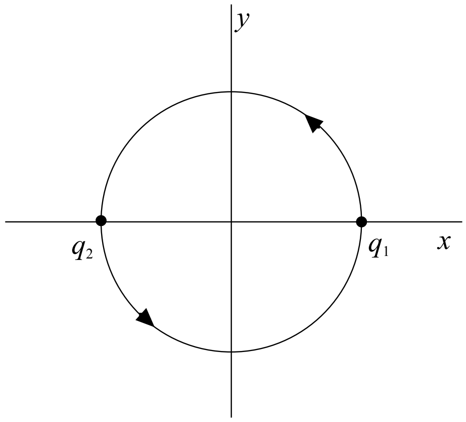
where $\omega_p \equiv \sqrt{4\pi ne^2/m}$ is the plasma frequency.

c) [10 points] For very low frequency waves such that $\omega \ll \omega_c$ and large magnetic fields such that $1/\tau \ll \omega_c$, show that $\omega = \omega_c(k^2c^2/\omega_p^2)$, i.e. $\omega \sim k^2$. Such waves are known as "helicons".

(turn over for problem #3)

3) [35 points total]

Consider two charges, q_1 and q_2, which are moving opposite to one another in a circular orbit of radius d with angular frequency ω in the xy plane, as sketched below.



a) [20 points] If $q_1 = -q_2 \equiv q$, find the radiated $\mathbf{E}$ and $\mathbf{B}$ fields in the electric dipole approximation. Express your answers as REAL functions of space and time. Find the time averaged Poynting vector as a function of space. Make a polar plot of the angular distribution of the radiated power, $dP/d\Omega$.

b) [15 points] What happens if one now has the case $q_1 = q_2 \equiv q$? What term in the multipole expansion is responsible for the radiation? What is the frequency of the radiation? You do not need to explicitly calculate $\mathbf{E}$, $\mathbf{B}$ or $\mathbf{S}$, but you must explain your reasoning clearly.

Hint: For a time dependent charge distribution with an oscillating electric dipole moment given by $\mathbf{p}(t) = \mathrm{Re}[\mathbf{p}_\omega \mathrm{e}^{-i\omega t)}]$, the radiated fields in the electric dipole approximation in the radiation zone are given by,

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[-k^2 \frac{\mathrm{e}^{i(kr-\omega t)}}{r} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{p}_\omega)\right] \quad , \quad \mathbf{B}(\mathbf{r},t) = \mathrm{Re}\left[k^2 \frac{\mathrm{e}^{i(kr-\omega t)}}{r} \hat{\mathbf{r}} \times \mathbf{p}_\omega\right] \quad \text{, where } k = \omega/c$$