Helmholtz Theorem

Suppose
$$\left.\begin{array}{l}\vec{\nabla}\cdot\vec{E}(\vec{r}) = f(\vec{r})\\ \vec{\nabla}\times\vec{E}(\vec{r}) = \vec{g}(\vec{r})\end{array}\right\}\text{ for } \vec{r} \text{ in a volume } V$$
$$\vec{E}(\vec{r}) = \vec{h}(\vec{r}) \quad \text{for } \vec{r} \text{ on surface } S \text{ of vol } V$$

Then if we know $f(\vec{r})$, $\vec{g}(\vec{r})$ and $\vec{h}(\vec{r})$, that information uniquely determines the vector function $\vec{E}(\vec{r})$

Proof:

Suppose we had two different solutions $\vec{E}(\vec{r})$ and $\vec{E}'(\vec{r})$

Then define
$$\vec{G}(\vec{r}) = \vec{E}(\vec{r}) - \vec{E}'(\vec{r})$$

$\vec{G}$ must satisfy
$$\left.\begin{array}{l}\vec{\nabla}\cdot\vec{G} = 0\\ \vec{\nabla}\times\vec{G} = 0\end{array}\right\}\text{ for all } \vec{r} \text{ in } V$$
$$\vec{G} = 0 \quad \text{for all } \vec{r} \text{ on } S$$

Now $\vec{\nabla}\times\vec{G} = 0$ implies we can find a scalar function ϕ such that $\vec{G} = \vec{\nabla}\phi$. Then
$$\vec{\nabla}\cdot\vec{G} = 0 \implies \nabla^2\phi = 0 \quad \text{for all } \vec{r} \text{ in } V.$$

A function ϕ that satisfies $\nabla^2\phi = 0$ within a region V is said to be a harmonic function on V.

An important property of harmonic functions is that the value at a position $\vec{r}$, is equal to the average of the values on the surface of a sphere centered at $\vec{r}$.

$$\phi(\vec{r}) = \frac{1}{4\pi R^2}\oint_{\tilde{S}} da'\, \phi(\vec{r}\,')$$

↑ surface of sphere of radius R centered at $\vec{r}$.

From this property we can conclude that a harmonic function on V can have no local maximum or minimum within the volume V. All maxima and minima must lie on surface S of V.

<u>Proof</u>: Just consider a small sphere centered on $\vec{r}$ that fits within the volume V. If $\vec{r}$ was a max, then for $\vec{r}\,'$ on surface of sphere, $\phi(\vec{r}\,') < \phi(\vec{r})$. But then we would have $\phi(\vec{r}) < \frac{1}{4\pi R^2} \oint da' \, \phi(\vec{r}\,')$ in violation of the above property of harmonic functions.

Back to our function $\vec{G}(\vec{r})$. We have

$\vec{\nabla} \cdot \vec{G} = 0$, $\vec{G} = \vec{\nabla}\phi \Rightarrow \nabla^2 \phi = 0$ in V

$\vec{G} = \vec{\nabla}\phi = 0$ on surface S of V $\Rightarrow \phi = $ constant on S.

All max and min of ϕ must be on surface S

$\Rightarrow \phi_{max} = \phi_{min} = $ constant.

$\Rightarrow \phi = $ constant throughout volume V

$\Rightarrow \vec{\nabla}\phi = \vec{G} = 0$ throughout V

$\Rightarrow \vec{E} = \vec{E}\,'$ for all $\vec{r}$ in V

$\Rightarrow$ solution is unique!

Magnetostatics

Lorentz Force

a charge q, in motion with velocity $\vec{v}$, feels the force

$$\vec{F} = q(\vec{E} + k_4 \, \vec{v} \times \vec{B}) \quad \leftarrow \text{Lorentz force}$$

$\vec{B}$ is the magnetic field at the position of the charge.
k_4 is a universal constant.

Just as the constant k_1 fixed the units of charge q, the constant k_4 can be viewed as fixing the units of B magnetic field. By choosing the units of q and B appropriately, we are free to choose any values for k_1 and k_4.

Magnetic field $\vec{B}$ is generated by moving charge
A charge q' with velocity $\vec{v}'$ ($v' \ll c$) located at the origin $\vec{r}'=0$ produces a magnetic field at position $\vec{r}$,

holds only nonrelativistically $\longrightarrow$

$$\vec{B}(\vec{r}) = k_5 \, q' \, \frac{\vec{v}' \times \vec{r}}{r^3} = \frac{k_5}{k_1} \, \vec{v}' \times \vec{E}(\vec{r})$$

k_5 is a universal constant. We will see that it cannot be chosen independently of k_1 and k_4. (since k_1 fixed units of q, and k_4 fixed units of $\vec{B}$, there are no further new quantities whose units could be adjusted to allow us to fix k_5 arbitrarily)

The force on a charge q at position $\vec{r}$, moving with velocity $\vec{v}$, due to a charge q' at the origin moving with velocity $\vec{v}'$ is, in non-relativistic limit ($v, v' \ll c$),

$$\vec{F} = \underbrace{k_1 q q' \frac{\vec{r}}{r^3}}_{\text{Coulomb force}} + \underbrace{k_4 k_5 q q' \, \vec{v} \times \frac{(\vec{v}' \times \vec{r})}{r^3}}_{\text{magnetic analog of Coulomb force}}$$

The magnetic part is just the point charge equivalent of the Biot-Savart law for the force between current ~~carry~~ carrying wires. If we regard $q\vec{v} = \vec{I}$ as the current of charge q, and $q'\vec{v}' = \vec{I}'$ as the current of charge q', then the magnetic force is $k_4 k_5 \, \vec{I} \times \left(\vec{I}' \times \frac{\vec{r}}{r^3}\right)$ which is the Biot-Savart Law.

Rewrite above force as

$$\vec{F} = k_1 \left(1 + \frac{k_4 k_5}{k_1} \, \vec{v} \times \vec{v}' \times \right) \frac{\vec{r}}{r^3} \, q q'$$

we see that $\left(\frac{k_4 k_5}{k_1}\right)$ has units of $(\text{velocity})^{-2}$

it must be independent of whatever convention one used to choose the units of q or B (ie independent of choices for k_1 and k_4). Experimentally it is found that

$$\left(\frac{k_4 k_5}{k_1}\right) = \frac{1}{c^2} \qquad c = \text{speed of light in vacuum}$$

Continuum current density

For charges q_i at positions $\vec{r}_i(t)$ with $\vec{v}_i = \frac{d\vec{r}_i}{dt}$ we define the current density

$$\vec{j}(\vec{r},t) = \sum_i q_i \vec{v}_i(t)\, \delta(\vec{r}-\vec{r}_i(t))$$

units of $\vec{j}$ are $(\text{charge})\left(\frac{\text{length}}{\text{time}}\right)\left(\frac{1}{\text{length}^3}\right) = \left(\frac{\text{charge}}{\text{area}\cdot\text{time}}\right)$

charge per unit area per unit time

For a surface S

$$\int_S da\, \hat{n}\cdot\vec{j} = I$$ current (charge per unit time) passing through surface S

Charge Conservation   vol V bounded by surface S

$$\frac{d}{dt}\int_V d^3r\, \rho(\vec{r},t) = -\oint_S da\, \hat{n}\cdot\vec{j}$$

rate of change of total charge in V $=$ $(-)$ charge flowing out of V through S per unit time

use $\oint_S da\,\hat{n}\cdot\vec{j} = \int_V d^3r\, \vec{\nabla}\cdot\vec{j} = -\int_V d^3r\, \frac{\partial}{\partial t}\rho(\vec{r},t)$

$\Rightarrow$ local charge conservation

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0}$$

A static situation has $\frac{\partial \rho}{\partial t} = 0$

$\Rightarrow$ magnetostatics is defined by the condition $\vec{\nabla}\cdot\vec{j} = 0$

Differential formulation of Biot-Savart

For a set of charges q_i at $\vec{r}_i$ we have

$$\vec{B}(\vec{r}) = \sum_i k_5\, q_i\, \vec{v}_i \times \frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3}$$

$$= k_5 \int d^3r'\, \vec{j}(\vec{r}\,') \times \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$$= k_5 \int d^3r'\, \vec{j}(\vec{r}\,') \times \vec{\nabla}\left(\frac{-1}{|\vec{r}-\vec{r}\,'|}\right)$$

$$\vec{B}(\vec{r}) = k_5\, \vec{\nabla}\times\left[\int d^3r'\, \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right]$$

where we used $\vec{\nabla}\times(\vec{A}\varphi) = -\vec{A}\times\vec{\nabla}\varphi$ when $\vec{A}$ is indep of $\vec{r}$

$\Rightarrow$ $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$ since $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ for any vector function $\vec{A}$

integral form $\oint_S da\, \hat{n}\cdot\vec{B} = 0$

$$\vec{\nabla}\times\vec{B} = k_5\, \vec{\nabla}\times\left[\vec{\nabla}\times\left(\int d^3r'\, \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right)\right]$$

use $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

$$\vec{\nabla}\times\vec{B} = k_5\,\vec{\nabla}\left[\int d^3r'\;\vec{\nabla}\cdot\left(\frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right)\right]$$

$$-k_5\int d^3r'\;\vec{j}(\vec{r}\,')\,\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$$

in the 2nd term, $\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}\,')$

in the 1st term, $\vec{\nabla}\cdot\frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} = \vec{j}(\vec{r}\,')\cdot\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$

$$= -\vec{j}(\vec{r}\,')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$$

since $\vec{\nabla} = -\vec{\nabla}'$

So $\int d^3r'\,\vec{\nabla}\cdot\left(\frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) = -\int d^3r'\,\vec{j}(\vec{r}\,')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$

integrate by parts
surface term $\to 0$ as
we take surface $\to\infty$
since $\vec{j}\to 0$ as $r\to\infty$

$$= \int d^3r'\left(\vec{\nabla}'\cdot\vec{j}(r')\right)\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$$

But for <u>magnetostatics</u> $\vec{\nabla}\cdot\vec{j} = 0 \Rightarrow$ only 2nd term remains

Thus, for <u>magnetostatics</u>

$$\vec{\nabla}\times\vec{B} = 4\pi k_5\,\vec{j} \qquad \text{Ampere's law}$$

integral form $\oint_C d\vec{l}\cdot\vec{B} = 4\pi k_5 \int_S da\,\hat{n}\cdot\vec{j}$

curve bounding surface

Although above diff eqns were derived starting from a "non-relativistic"