i) Lorentz Gauge

gauge constraint: require $\frac{1}{c}\frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$

Then Gauss' Law becomes

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -4\pi\rho$$

$$\Rightarrow \nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right) = -4\pi\rho$$

$$\boxed{\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho}$$

Ampere's Law becomes

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{J} - \vec{\nabla}\underbrace{\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)}_{\overset{\shortparallel}{0}}$$

$$\boxed{\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J}}$$

The combination $-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \equiv \Box^2$ is the <u>wave equation</u> operator.

In Lorentz gauge, $\vec{A}$ and ϕ satisfy the inhomogeneous wave equations:

$$\boxed{\begin{aligned}\Box^2\vec{A} &= \frac{4\pi}{c}\vec{J}\\ \Box^2\phi &= 4\pi\rho\end{aligned}}$$

when $\vec{J}=0$, $\rho=0$ electromagnetic waves are solution!

proof that we can always find $\vec{A}$ and ϕ that satisfy the Lorentz gauge condition

Suppose $\vec{\nabla}\times\vec{A} = \vec{B}$ and $-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \vec{E}$

but $\frac{1}{c}\frac{\partial \phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = D(\vec{r},t) \neq 0$

Construct $\vec{A}' = \vec{A} + \vec{\nabla}\chi$

$\phi' = \phi - \frac{1}{c}\frac{\partial \chi}{\partial t}$

by gauge invariance we know $\vec{A}'$ and ϕ' give the same $\vec{E}$ and $\vec{B}$ as before.

now: $\vec{\nabla}\cdot\vec{A}' + \frac{1}{c}\frac{\partial \phi'}{\partial t} = \vec{\nabla}\cdot\vec{A} + \nabla^2\chi + \frac{1}{c}\frac{\partial \phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2}$

$= D - \Box^2\chi$

So $\vec{A}'$ and ϕ' will be in the Lorentz gauge provided we choose $\chi(\vec{r},t)$ such that

$$\Box^2\chi = D \quad \leftarrow \text{inhomogeneous wave equation}$$

Just like there is always a solution to Poisson's Eq $\nabla^2\phi = f$, so there is always a solution to the inhomogeneous wave equation, hence we can always find a $\chi(\vec{r},t)$ that transforms to the Lorentz gauge

Note: Lorentz gauge condition does *not* uniquely determine $\vec{A}$ and ϕ. If one ~~constructs~~ has $\vec{A}$ and ϕ obeying Lorentz gauge condition, and then constructs

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi$$

$$\phi' = \phi - \frac{1}{c}\frac{\partial \chi}{\partial t}$$

Then $\vec{A}'$ and ϕ' will also be in Lorentz gauge *provided* $\Box^2 \chi = 0$ (proof left to reader)

2) Coulomb Gauge

gauge constraint: require $\vec{\nabla}\cdot\vec{A} = 0$
if $\vec{A}$ is in the Coulomb Gauge, then
$\vec{A}' = \vec{A} + \vec{\nabla}\chi$ will also be in Coulomb gauge
provided $\nabla^2 \chi = 0$.

Then Gauss' law becomes

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -4\pi\rho$$

$$\Rightarrow \boxed{\nabla^2\phi = -4\pi\rho}$$ same as electrostatics!

$$\Rightarrow \phi(\vec{r},t) = \int d^3r' \, \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$

no matter what motion the source $\rho(\vec{r},t)$ has! ϕ is given by the instantaneous Coulomb potential even though electromagnetic fields have a finite velocity of propagation c!

Ampere's Law becomes:

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{j} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)$$

$$\Box^2\vec{A} = \frac{4\pi}{c}\vec{j} - \frac{1}{c}\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) \qquad \text{since } \vec{\nabla}\cdot\vec{A}=0$$

Now use the solution for ϕ in the Coulomb gauge to write

$$\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) = \vec{\nabla}\left[\int d^3r' \,\frac{\partial\rho(\vec{r}',t)}{\partial t}\,\frac{1}{|\vec{r}-\vec{r}'|}\right]$$

$$= -\vec{\nabla}\left[\int d^3r' \,\frac{\vec{\nabla}'\cdot\vec{j}(\vec{r}',t)}{|\vec{r}-\vec{r}'|}\right]$$

last step follows from conservation of charge $\vec{\nabla}\cdot\vec{j} = -\frac{\partial\rho}{\partial t}$

To see the meaning of this term, recall (and we will soon demonstrate explicitly) that <u>any</u> vector function $\vec{f}(\vec{r},t)$ can always be written as the sum of a curlfree part and a divergenceless part

$$\vec{f} = \vec{f}_{\parallel} + \vec{f}_{\perp} \quad \text{where} \quad \vec{\nabla}\times\vec{f}_{\parallel} = 0 \ \text{ curlfree}$$
$$\vec{\nabla}\cdot\vec{f}_{\perp} = 0 \ \text{ divergenceless}$$

when $\vec{\nabla}\cdot\vec{f}$ and $\vec{\nabla}\times\vec{f}$ are localized functions that vanish as $\vec{r}\to\infty$, we have for solutions (proof to follow)

$$\vec{f}_{\parallel}(\vec{r}) = -\frac{1}{4\pi}\vec{\nabla}\int d^3r' \,\frac{\vec{\nabla}'\cdot\vec{f}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\vec{f}_{\perp}(\vec{r}) = \frac{1}{4\pi}\vec{\nabla}\times\int d^3r' \,\frac{\vec{\nabla}'\times\vec{f}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

The *curlfree* part is also called the *longitudinal* part
the *divergenceless* part is also called the *transverse* part

Returning to Ampere's law we see that the term

$$\vec\nabla\left(\frac{\partial\phi}{\partial t}\right) = -\vec\nabla \int d^3r' \left[\frac{\vec\nabla'\cdot\vec j(r',t)}{|\vec r-\vec r'|}\right]$$

$$= 4\pi\,\vec j_{\parallel}(\vec r,t)$$

So Amperes Law becomes

$$\Box^2\vec A = \frac{4\pi}{c}\vec j - \frac{4\pi}{c}\vec j_{\parallel}$$

$$\boxed{\Box^2\vec A = \frac{4\pi}{c}\vec j_{\perp}}$$

In Coulomb gauge, only the transverse part of $\vec j$ serves as a source for $\vec A$.

$\vec A$ describes the *transverse* modes, ie the EM radiation (recall in EM waves, the fields are always ⊥ direction of propagation)

ϕ describes the *longitudinal* modes

Coulomb gauge is *not* Lorentz invariant - if $\vec\nabla\cdot\vec A=0$ in one inertial reference frame, in general $\vec\nabla\cdot\vec A\neq0$ in another.

In Coulomb gauge, if $\rho=0$, then $\phi=0$ and

$$\vec E = -\frac{1}{c}\frac{\partial\vec A}{\partial t}$$

Transverse + Longitudinal Parts of vector functions

To prove the preceeding claim, $\vec{j} = \vec{j}_\parallel + \vec{j}_\perp$, where $\vec{\nabla}\times\vec{j}_\parallel = 0$ and $\vec{\nabla}\cdot\vec{j}_\perp = 0$, we first digress to prove Helmholtz theorem.

Helmholtz Theorem: For a vector function $\vec{f}(\vec{r})$ if one knows the divergence and curl of $\vec{f}$ then one can ~~uniquely~~ uniquely determine $\vec{f}$ itself.

That is, if

$\vec{\nabla}\cdot\vec{f} = 4\pi D(\vec{r})$ where $D(\vec{r})$ is a known scalar function

$\vec{\nabla}\times\vec{f} = 4\pi \vec{C}(\vec{r})$ where $\vec{C}(\vec{r})$ is a known vector function

~~Then one can solve for~~

And if well defined boundary conditions on $\vec{f}$ are known (here we will assume $\vec{f}(\vec{r})\to 0$ as $\vec{r}\to\infty$) Then there is a unique solution for $\vec{f}(\vec{r})$.

We prove this by construction!

Assume a solution of the form

$\vec{f} = -\vec{\nabla}\varphi + \vec{\nabla}\times\vec{W}$ where φ is a scalar and $\vec{W}$ a vector

Now we show that we can find such a solution

First consider

$$\vec{\nabla}\cdot\vec{f} = -\nabla^2\varphi + \vec{\nabla}\cdot(\vec{\nabla}\times\vec{W}) = -\nabla^2\varphi + 0 = 4\pi D(\vec{r})$$

So $-\nabla^2\varphi = 4\pi D(\vec{r})$ This is just Poisson's equ we saw in electrostatics

Solution when $\varphi(\vec{r})\to 0$ as $\vec{r}\to\infty$ is given by

$$\boxed{\varphi(\vec{r}) = \int d^3r' \, \frac{D(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}}$$

Coulomb-like integral solution

Now consider

$$\vec{\nabla}\times\vec{f} = -\vec{\nabla}\times\vec{\nabla}\varphi + \vec{\nabla}\times(\vec{\nabla}\times\vec{W}) = 0 - \nabla^2\vec{W} + \vec{\nabla}(\vec{\nabla}\cdot\vec{W})$$
$$= 4\pi\vec{C}(\vec{r})$$

Choose a gauge in which $\vec{\nabla}\cdot\vec{W}=0$ (just like Coulomb gauge in magnetostatics)

Then $-\nabla^2\vec{W} = 4\pi\vec{C}(\vec{r})$

$$\boxed{\vec{W}(\vec{r}) = \int d^3r' \, \frac{\vec{C}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}}$$

just like solution for vector pot $\vec{A}$ in magnetostatics

So we have constructed a solution

$$f(\vec{r}) = -\vec{\nabla}\varphi + \vec{\nabla}\times\vec{W}$$

$$= -\vec{\nabla}\int d^3r' \, \frac{D(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \vec{\nabla}\times\int d^3r' \, \frac{\vec{C}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

where $\vec{\nabla}\cdot\vec{f} = 4\pi D$ and $\vec{\nabla}\times\vec{f} = 4\pi\vec{C}$

Note: For above solution to be well defined, the integrals must converge. They will converge if the "sources" $D(\vec{r})$ and $\vec{C}(\vec{r})$ are sufficiently "localized" in space, ie $D(\vec{r}) \to 0$, $\vec{C}(\vec{r}) \to 0$ sufficiently fast as $\vec{r} \to \infty$.

Now we show that the above solution is <u>unique</u>.

Suppose there was another solution $\vec{g}$ such that

$$\vec{\nabla} \cdot \vec{g} = 4\pi D \quad \text{and} \quad \vec{\nabla} \times \vec{g} = 4\pi \vec{C}$$

Consider $\vec{h} \equiv \vec{f} - \vec{g}$ then

$$\vec{\nabla} \cdot \vec{h} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{h} = 0$$

Can show that only such $\vec{h}$ that also has $\vec{h}(\vec{r}) \to 0$ as $\vec{r} \to \infty$ is $\vec{h} \equiv 0$, so $\vec{g} = \vec{f}$ and solution is unique.

As a consequence of Helmholtz theorem we have also shown the following

① Any vector function $\vec{f}$ can be written in terms of a scalar and vector potential

$$\vec{f} = -\vec{\nabla}\varphi + \vec{\nabla} \times \vec{W}$$

or equivalently

② Any vector function $\vec f$ can be written in terms of a curl free and a divergenceless part

$$\vec f = \vec f_{\parallel} + \vec f_{\perp} \quad \text{where} \quad \vec\nabla \times f_{\parallel} = 0 \quad \text{curlfree}$$
$$\vec\nabla \cdot \vec f_{\perp} = 0 \quad \text{divergenceless}$$

$$\text{where} \begin{cases} \vec f_{\parallel}(\vec r) = -\vec\nabla \varphi(\vec r) = -\vec\nabla \int \frac{d^3r'}{4\pi} \frac{[\vec\nabla' \cdot \vec f(\vec r')]}{|\vec r - \vec r'|} \\ \vec f_{\perp}(\vec r) = \vec\nabla \times \vec W(\vec r) = \vec\nabla \times \int \frac{d^3r'}{4\pi} \frac{[\vec\nabla' \times \vec f(\vec r')]}{|\vec r - \vec r'|} \end{cases}$$

where in above we used $\vec D(\vec r') = \frac{1}{4\pi} \vec\nabla' \cdot f(\vec r')$

$$\vec C(\vec r') = \frac{1}{4\pi} \vec\nabla' \times \vec f(\vec r')$$

~~where~~ $\vec f_{\parallel}$ is called the longitudial part of $\vec f$

$\vec f_{\perp}$ is called the transverse part of $\vec f$

To understand the reason for these names, we need to consider the Fourier transforms

Above can be generalized to situations where $\vec f$ satisfies other boundary conditions, say has a specified value on a given boundary surface. One just replaces $\frac{1}{|\vec r - \vec r'|}$ by the appropriate Greens function — see more to come!

Digression regarding Fourier transforms

$$\vec{f}(\vec{r}) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \vec{f}(\vec{k})$$ Fourier transf

$$\vec{f}(\vec{k}) = \int_{-\infty}^{\infty} d^3r\, e^{-i\vec{k}\cdot\vec{r}} \vec{f}(\vec{r})$$ inverse transf

Some special cases well worth remembering

① Transform of Dirac function

$$\int d^3r\, e^{-i\vec{k}\cdot\vec{r}} \delta(\vec{r}-\vec{r}_0) = e^{-i\vec{k}\cdot\vec{r}_0}$$

$$\Rightarrow \delta(\vec{r}-\vec{r}_0) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} e^{-i\vec{k}\cdot\vec{r}_0}$$

$$\boxed{\delta(\vec{r}-\vec{r}_0) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{r}-\vec{r}_0)}}$$

or letting $\vec{r} \leftrightarrow \vec{k}$ in the above

$$\delta(\vec{k}-\vec{k}_0) = \int \frac{d^3r}{(2\pi)^3} e^{i\vec{r}\cdot(\vec{k}-\vec{k}_0)}$$

② Transform of Coulomb potential $\frac{1}{|\vec{r}-\vec{r}'|}$

We know

$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}')$$

Suppose $f(\vec{k}) \equiv \int_{-\infty}^{\infty} d^3r\, e^{-i\vec{k}\cdot\vec{r}} \frac{1}{|\vec{r}-\vec{r}'|}$ is the

Fourier transf of $\frac{1}{|\vec{r}-\vec{r}'|}$

Substitute
$$\begin{cases} \dfrac{1}{|\vec{r}-\vec{r}'|} = \displaystyle\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, f(\vec{k}) \\ \delta(\vec{r}-\vec{r}') = \displaystyle\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \end{cases}$$

into above Poisson equation

$$\underbrace{\nabla^2 \int \frac{d^3k}{(2\pi)^3}}_{\text{operates only on } \vec{r}} e^{i\vec{k}\cdot\vec{r}} f(\vec{k}) = -4\pi \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

operates only on $\vec{r}$
so move inside integral

$$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = \vec{\nabla}\cdot\left(\vec{\nabla} e^{i\vec{k}\cdot\vec{r}}\right)$$

① $$\vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = \sum_{i=1}^{3} \hat{x}_i \frac{\partial}{\partial x_i} e^{i\vec{k}\cdot\vec{r}} = \sum_{i=1}^{3} \hat{x}_i\, i k_i\, e^{i\vec{k}\cdot\vec{r}}$$

$$= i\vec{k}\, e^{i\vec{k}\cdot\vec{r}}$$ where $\hat{x}_1, \hat{x}_2, \hat{x}_3 = \hat{x}, \hat{y}, \hat{z}$

② $$\vec{\nabla}\cdot\left(i\vec{k}\, e^{i\vec{k}\cdot\vec{r}}\right) = (i\vec{k})\cdot(i\vec{k})\, e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

so $$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

Poisson eqn then gives

$$\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} (-k^2) f(\vec{k}) = -4\pi \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, e^{-i\vec{k}\cdot\vec{r}'}$$

$$\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} \left[-k^2 f(\vec{k})\right] = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} \left[-4\pi e^{-i\vec{k}\cdot\vec{r}'}\right]$$

As is true for Fourier series, so it is true for Fourier transforms: If two functions are equal, then their Fourier transforms are equal.

$$\Rightarrow \quad -k^2 f(\vec{k}) = -4\pi e^{-i\vec{k}\cdot\vec{r}'}$$

$$\boxed{f(\vec{k}) = \frac{4\pi}{k^2} e^{-i\vec{k}\cdot\vec{r}'}}$$

$\Rightarrow$ is the Fourier transform of $\frac{1}{|\vec{r}-\vec{r}'|}$