Green's theorem, Uniqueness, Green function - part II

We want to show that the boundary value problem we described is well posed - ie there is a unique solution. We start by deriving Greens theorems

Consider $\int_V d^3r \, \vec{\nabla}\cdot\vec{A} = \oint_S da \, \hat{n}\cdot\vec{A}$ Gauss theorem

let $\vec{A} = \phi \vec{\nabla}\psi$ ϕ, ψ any two scalar functions

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{A} = \phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi$$

$$\phi\vec{\nabla}\psi\cdot\hat{n} = \phi\frac{\partial\psi}{\partial n}$$

$$\Rightarrow \int_V d^3r \left(\phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi\right) = \oint_S da \, \phi\frac{\partial\psi}{\partial n} \quad \Big\} \text{ Green's 1st - identity}$$

let $\phi \leftrightarrow \psi$

$$\int_V d^3r \left(\psi\nabla^2\phi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi\right) = \oint_S da \, \psi\frac{\partial\phi}{\partial n}$$

subtract

$$\int_V d^3r \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) = \oint_S da \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) \quad \Big\} \text{ Green's 2nd identity}$$

Apply Greens 2nd identity with $\psi = \frac{1}{|\vec{r}-\vec{r}'|}$, $\vec{r}'$ is integration variable, ϕ is the scalar potential with $\nabla^2\phi = -4\pi\rho$. Use $\nabla^2\psi = \nabla'^2\psi = -4\pi\,\delta(\vec{r}-\vec{r}')$

$$\int_V d^3r' \left[\phi(\vec{r}')\left[-4\pi\,\delta(\vec{r}-\vec{r}')\right] - \left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\left(-4\pi\rho(\vec{r}')\right)\right]$$

$$= \oint_S da' \left[\phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) - \frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi}{\partial n'}\right]$$

If $\vec{r}$ lies within the volume V, then

$$(*) \qquad \phi(\vec{r}) = \int_V d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint_S \frac{da'}{4\pi}\left[\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right]$$

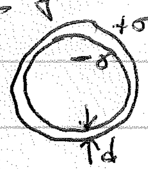
Note: if $\vec{r}$ lies outside the volume V, then

$$(**) \qquad 0 = \int_V d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint \frac{da'}{4\pi}\left[\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right]$$

potential from a surface charge density $\sigma = \frac{1}{4\pi}\frac{\partial\phi}{\partial n'}$

potential from a surface dipole layer of dipole strength density $\frac{\phi}{4\pi}$

dipole layer:



$d \to 0$ such that σd stays finite

From (*), if $S \to \infty$ and $E \sim \frac{\partial\phi}{\partial n} \to 0$ faster than $\frac{1}{r}$, then the surface integral vanishes and we recover Coulombs law $\phi(\vec{r}) = \int d^3r' \, \rho(\vec{r}')/|\vec{r}-\vec{r}'|$

(*) gives the generalization of Coulombs law to a system with a finite boundary

For a charge free volume V, ie $\rho(r) = 0$ in V, the potential everywhere is determined by the potential and its normal derivative on the surface.

But one cannot in general freely specify both ϕ and $\frac{\partial\phi}{\partial n}$ on the boundary surface since the resulting ϕ from (*) would not in general obey Laplace's equation $\nabla^2\phi = 0$, nor would (**) vanish.

Specifying both ϕ and $\frac{\partial \phi}{\partial n}$ on surface is known as "Cauchy" boundary conditions — for Laplace's eqn, Cauchy b.c. overspecify the problem + a solution cannot in general be found.

Uniqueness

If we have a system of charges in vol V, and either the potential ϕ, or its normal derivative $\frac{\partial \phi}{\partial n}$, is specified on the surfaces of V, then there is a unique solution to Poisson's equation inside V. Specifying ϕ is known as Dirichlet boundary conditions. Specifying $\frac{\partial \phi}{\partial n}$ is known as Neumann boundary conditions.

proof: Suppose we had two solutions ϕ_1 and ϕ_2, both with $-\nabla^2\phi = 4\pi\rho$ inside V, and obeying specified b.c. on surface of V.

Define $u = \phi_2 - \phi_1 \;\Rightarrow\; \nabla^2 u = 0$ inside V

and $u = 0$ on surface S - for Dirichlet b.c.
or $\frac{\partial u}{\partial n} = 0$ on surface S - for Neumann b.c.

Use Green's 1st identity with $\phi = \psi = u$

$$\int_V d^3r \, (\underbrace{u\nabla^2 u}_{\substack{\parallel \\ 0 \\ \text{as } \nabla^2 u = 0}} + \vec{\nabla}u \cdot \vec{\nabla}u) = \oint_S da \, \underbrace{u \frac{\partial u}{\partial n}}_{\substack{\parallel \\ 0 \\ \text{as } u \text{ or } \frac{\partial u}{\partial n} = 0}}$$

$$\Rightarrow \int_V d^3r \, |\vec{\nabla} U|^2 = 0 \qquad \Rightarrow \vec{\nabla} U = 0$$
$$\Rightarrow U = \text{const}$$

For Dirichlet b.c., $U = 0$ on surface S, so const $= 0$ and $\phi_1 = \phi_2$. Solution is unique

For Neumann b.c., ϕ_1 and ϕ_2 differ only by an arbitrary constant. Since $\vec{E} = -\vec{\nabla}\phi$, the electric fields $\vec{E}_1 = -\vec{\nabla}\phi_1$ and $E_2 = -\vec{\nabla}\phi_2$ are the same.

~~Solution~~ If boundary ~~solution~~ surface S consists of several disjoint pieces, then solution is unique if specify ϕ on some pieces and $\frac{\partial \phi}{\partial n}$ on other pieces.

Solution of Poisson's equation with both ϕ and $\frac{\partial \phi}{\partial n}$ specified on the same surface S (Cauchy b.c.) does not in general exist, since specifying either ϕ or $\frac{\partial \phi}{\partial n}$ alone is enough to give a unique solution.

Green's function - part II

Greens 2^{nd} identity

$$\int_V d^3r' \left(\phi \nabla'^2 \psi - \psi \nabla'^2 \phi \right) = \oint_S da' \left(\phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'} \right)$$

Apply above with $\begin{cases} \phi(\vec{r}') \text{ electrostatic potential with } \nabla'^2\phi = -4\pi\rho(\vec{r}') \\ \psi(\vec{r}') = G(\vec{r},\vec{r}') \text{ the Green function satisfying} \end{cases}$

$$\nabla'^2 G(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$$

we saw one solution of above is $G(\vec{r},\vec{r}') = \dfrac{1}{|\vec{r}-\vec{r}'|}$

but a more general solution is

$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} + F(\vec{r},\vec{r}')$$

where $\nabla'^2 F(\vec{r},\vec{r}') = 0$, for $\vec{r}'$ in volume V

we will choose $F(\vec{r},\vec{r}')$ to simplify solution of ϕ

$$\Rightarrow \int_V d^3r' \left(\phi(\vec{r}')\, \nabla'^2 G(\vec{r},\vec{r}') - G(\vec{r},\vec{r}')\, \nabla'^2 \phi(\vec{r}') \right)$$

$$= \int_V d^3r' \left(\phi(\vec{r}') \left[-4\pi\delta(\vec{r}-\vec{r}')\right] - G(\vec{r},\vec{r}') \left[-4\pi\rho(\vec{r}')\right] \right)$$

$$= -4\pi\,\phi(\vec{r}) + 4\pi \int_V d^3r'\, G(\vec{r},\vec{r}')\, \rho(\vec{r}')$$

<u>for $\vec{r}$ in V</u>

$$= \oint_S da' \left(\phi \frac{\partial G}{\partial n'} - G \frac{\partial \phi}{\partial n'} \right)$$

$$\phi(\vec{r}) = \int_V d^3r' \, G(\vec{r},\vec{r}') \, \rho(\vec{r}') + \oint_S \frac{da'}{4\pi} \left(G(\vec{r},\vec{r}') \frac{\partial \phi(\vec{r}')}{\partial n'} - \phi(r') \frac{\partial G(\vec{r},\vec{r}')}{\partial n'} \right)$$

Consider <u>Dirichlet boundary problem</u>. If we can choose $F(\vec{r},\vec{r}')$ such that $G_D(\vec{r},\vec{r}') = 0$ for $\vec{r}'$ on the boundary surface S, then above simplifies to

$$\boxed{\phi(\vec{r}) = \int_V d^3r' \, G_D(\vec{r},\vec{r}') \, \rho(\vec{r}') - \oint_S \frac{da'}{4\pi} \phi(r') \frac{\partial G_D(\vec{r},\vec{r}')}{\partial n'}}$$

Since $\rho(r)$ is specified in V, and $\phi(r)$ is specified on S, above then gives desired solution ~~of~~ for $\phi(r)$ inside volume V.

Finding G_D is therefore equivalent to finding an $F(\vec{r},\vec{r}')$ such that $\nabla'^2 F(\vec{r},\vec{r}') \overset{?}{=} 0$ for $\vec{r}'$ in V (solves Laplace eqn) and

$$F(\vec{r},\vec{r}') = \frac{-1}{|\vec{r}-\vec{r}'|} \quad \text{for } \vec{r}' \text{ on boundary surface } S'$$

Always exists unique solution for F

Next consider <u>Neumann boundary problem</u>.

One might think to find $F(\vec{r},\vec{r}\,')$ such that $\frac{\partial G(\vec{r},\vec{r}\,')}{\partial n'} = 0$ on boundary surface. But this is not possible.

Consider $\int_V \nabla'^2 G(\vec{r},\vec{r}\,')\, d^3r' = \int \vec{\nabla}' \cdot \vec{\nabla}' G(\vec{r},\vec{r}\,')\, d^3r'$

$$= \oint_S \vec{\nabla}' G(\vec{r},\vec{r}\,') \cdot \hat{n}\, da'$$

$$= \oint_S \frac{\partial G(\vec{r},\vec{r}\,')}{\partial n'}\, da' = -4\pi \quad \text{since } \nabla'^2 G = -4\pi\delta(r-r')$$

So we can't have $\frac{\partial G}{\partial n'} = 0$ for $\vec{r}\,'$ on S

Simplest choice is then $\frac{\partial G_N(r,r')}{\partial n'} = -\frac{4\pi}{S}$ for $\vec{r}\,'$ on S

area of surface

Then

$$\phi(\vec{r}) = \int_V d^3r'\, G_N(\vec{r},\vec{r}\,')\, \rho(\vec{r}\,') + \oint \frac{da'}{4\pi}\, G_N(\vec{r},\vec{r}\,')\, \frac{\partial \phi(\vec{r}\,')}{\partial n'}$$

$$- \oint \frac{da'}{4\pi}\, \phi(r') \left(\frac{-4\pi}{S}\right)$$

$$\left[\phi(\vec{r}) = \int_V d^3r'\, G_N(\vec{r},\vec{r}\,')\, \rho(\vec{r}\,') + \oint \frac{da'}{4\pi}\, G_N(\vec{r},\vec{r}\,')\, \frac{\partial \phi(\vec{r}\,')}{\partial n'}\right]$$

$$+ \langle \phi \rangle_S$$

constant = average value of ϕ on surface S.

Since $\rho(r)$ is specified in V and $\frac{\partial \phi}{\partial n}$ is specified on S

above gives solution $\phi(r)$ in V within additive constant $\langle \phi \rangle_S$

Since $\vec{E} = -\vec{\nabla}\phi$, the const $\langle \phi \rangle_S$ is of no consequence

Finding $G_N(\vec{r},\vec{r}')$ is therefore equivalent to finding an $F(\vec{r},\vec{r}')$ such that

$$\nabla'^2 F(\vec{r},\vec{r}') = 0 \quad \text{for } \vec{r}' \text{ in } V$$

$$\text{and} \quad \frac{\partial F(\vec{r},\vec{r}')}{\partial n'} = \frac{-4\pi}{S'} \quad \text{for } \vec{r}' \text{ on surface } S'$$

always exists a unique solution (within additive constant)

While G_D and G_N always exist in principle, they depend in detail on the shape of the surface S' and are difficult to find except for simple geometries

In preceding we defined G by $\nabla'^2 G(\vec{r},\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}')$

But our earlier interpretation of $G(\vec{r},\vec{r}')$ was that it was potential at $\vec{r}$ due to point source at $\vec{r}'$, ie $\nabla^2 G(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$. Note, for general surface S', $G(\vec{r},\vec{r}')$ is not in general a function of $|\vec{r}-\vec{r}'|$ but depends on $\vec{r}$ and $\vec{r}'$ separately. But the equivalence of the two definitions of G above is obtained by noting that one can prove the symmetry property

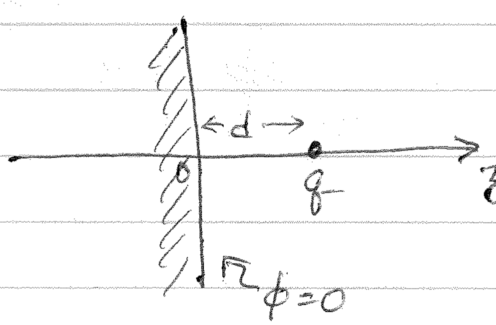
$$G(\vec{r},\vec{r}') = G(\vec{r}',\vec{r})$$

for Dirichlet b.c., and one can impose it as an additional requirement for Neumann b.c. (see Jackson, end section 1.10)

Image Charge method

For simple geometries, can try to obtain G_D or G_N by placing a set of "image charges" outside the volume of interest V, ie on the "other side" of the system boundary surface S. Because these image charges are outside V, their contrib to the potential inside V obeys $\nabla^2\phi^{image} = 0$, as necessary. Choose location of image charges so that total ϕ has desired boundary condition.

1) charge in front of infinite grounded plane



want $\nabla^2\phi = -4\pi q\,\delta(x)\delta(y)\delta(z-d)$

$\phi = 0$ for $z=0$

If we find a solution to above it is the unique solution

Solution - put fictitious image charge $-q$ at $z=-d$

ϕ is Coulomb potential from the real charge + the image

$$\phi(\vec r) = \underbrace{\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}}_{\text{real charge}} + \underbrace{\frac{-q}{\sqrt{x^2+y^2+(z+d)^2}}}_{\text{image charge}}$$

above satisfies $\phi(x,y,0) = 0$ as required

Also, $\nabla^2\phi = -4\pi q\,\delta(\vec r - d\hat z) + 4\pi q\,\delta(\vec r + d\hat z)$

$= -4\pi q\,\delta(\vec r - d\hat z)$ for region $z>0$

Can now find $\vec{E}$ for $z > 0$

$$\vec{E} = -\vec{\nabla}\phi$$

In particular
$$E_z = -\frac{\partial \phi}{\partial z} = +q\left[\left(\tfrac{1}{2}\right)\frac{2(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \left(\tfrac{1}{2}\right)\frac{2(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$$

$$E_z = q\left[\frac{(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$$

We can use above to compute the surface charge density $\sigma(x,y)$ induced on the surface of the conducting plane. At conductor surface

$$-\frac{\partial \phi}{\partial n} = 4\pi\sigma$$

since in general $-\frac{\partial \phi}{\partial n}^{\text{above}} + \frac{\partial \phi}{\partial n}^{\text{below}} = 4\pi\sigma$
and for a conductor $\frac{\partial \phi}{\partial n}^{\text{below}} = 0$

$$\Rightarrow \quad \sigma = -\frac{1}{4\pi}\frac{\partial \phi}{\partial z} = \frac{1}{4\pi} E_z(x,y,z=0)$$

$$\sigma(x,y) = \frac{q}{4\pi}\left[\frac{-d}{(x^2+y^2+d^2)^{3/2}} - \frac{d}{(x^2+y^2+d^2)^{3/2}}\right]$$

$$= -\frac{q}{2\pi}\frac{d}{(x^2+y^2+d^2)^{3/2}} = \frac{-qd}{2\pi\,(r_\perp^2+d^2)^{3/2}}$$

$$r_\perp = \sqrt{x^2+y^2}$$