$$\phi(\vec{r}) = \frac{q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \frac{\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{2r^3} + \ldots$$

Note, in each term the dependence on $r = |\vec{r}|$ is only in the denominator. The numerators depend on the orientation of $\vec{r}$ via $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$. So we can write

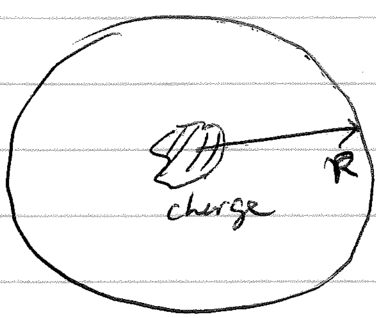
$$\phi(\vec{r}) = \frac{q}{r} + \sum_{n=2}^{\infty} \frac{f_n(\theta,\varphi)}{r^n}$$

where $f_n(\theta,\varphi)$ gives the dependence on the orientation of $\vec{r}$

Now we show that the angular average of $f_n(\theta,\varphi)$ must vanish, ie $\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi \, f_n(\theta,\varphi) = 0$

Consider the corresponding electric field $\vec{E} = -\vec{\nabla}\phi$

$$\vec{E} = q\frac{\hat{r}}{r^2} + \sum_{n=2}^{\infty} \vec{\nabla}\left(\frac{f_n(\theta,\varphi)}{r^n}\right)$$



consider a sphere of radius R centered on the charge distribution. Let S be the surface of this sphere. We know that

$$\oint_S da\, \hat{n}\cdot\vec{E} = 4\pi Q_{enclosed} = 4\pi q \leftarrow \text{monopole moment}$$

Since $\oint_S da\, \hat{n}\cdot q\frac{\hat{r}}{r^2} = \oint_S da \frac{q}{r^2}$ since $\hat{n} = \hat{r}$

$$= 4\pi R^2 \frac{q}{R^2} = 4\pi q$$

So monopole term gives all the flux of $\vec{E}$ through the surface, and the flux of the higher terms must give zero. Since this must be true for any radius R, it can only be true if each term individually gives zero flux, i.e.

$$-\oint_S da\ \hat{r}\cdot\vec{\nabla}\left(\frac{f_n(\theta,\varphi)}{r^n}\right) = 0$$

but $\hat{r}\cdot\vec{\nabla} = \frac{\partial}{\partial r}$ radial directional derivative

so above is

$$-\oint_S da\ \frac{\partial}{\partial r}\left(\frac{f_n(\theta,\varphi)}{r^n}\right) = n\oint_S da\ \frac{f_n(\theta,\varphi)}{r^{n+1}}$$

$$= \frac{n}{R^{n+1}}\oint_S da\ f_n(\theta,\varphi) \qquad \text{since } r=R \text{ on } S$$

$$= \frac{n}{R^{n+1}} R^2\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi\ f_n(\theta,\varphi) = 0$$

$$\text{so} \quad \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi\ f_n(\theta,\varphi) = 0$$

So the n-th moment contribution to Φ

$\Phi^{(n)} = \dfrac{f_n(\theta,\varphi)}{r^n}$ vanishes if we take an angular average.

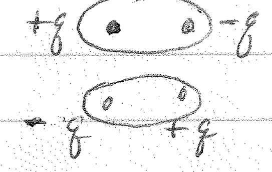
Example

sample charge config's

$\circ\ q$ $\Rightarrow$ monopole is leading term

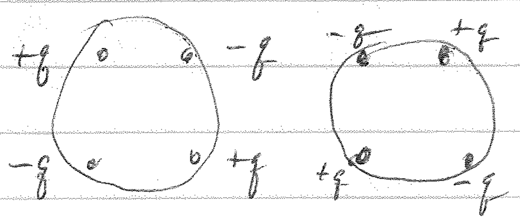
$\overset{\circ}{+q}\ \ \overset{\circ}{-q}$ $\Rightarrow$ monopole $= 0$ $\Rightarrow$ dipole is leading term

$\vec{p}$ is indep of origin



$\Rightarrow$ monopole $= 0$ $\Rightarrow$ total dipole is sum of dipoles of individual neutral pairs

$= 0$ leading term is quadrapole



when monopole $= 0$ and dipole $= 0$, quadrapole is indep of origin.

$\rightarrow$ total quadrapole is sum of quadrapoles of individual clusters with $q = 0$ and $\vec{p} = 0$

$Q = Q_1 + Q_2$

with $Q_2 = -Q_1$

$\Rightarrow Q = 0$ leading term is <u>octopole</u>

Magnetostatics

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad , \quad \begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} \end{cases} \quad \text{Ampere's law (statics only!)}$$

$$\Rightarrow \quad \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{4\pi}{c}\vec{j}$$

can write $\quad \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

where by $\nabla^2\vec{A}$ we mean $(\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z}$

$\nabla^2\vec{A}$ only has a simple expression in Cartesian coords

Aside

If tried to write it in spherical coords, for example, one has

$$\nabla^2\vec{A} = \nabla^2(A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi})$$

$$= (\nabla^2 A_r)\hat{r} + A_r(\nabla^2\hat{r}) + (\nabla^2 A_\theta)\hat{\theta} + A_\theta(\nabla^2\hat{\theta}) + (\nabla^2 A_\varphi)\hat{\varphi} + A_\varphi(\nabla^2\hat{\varphi})$$

one must not forget to take the derivates of $\hat{r}, \hat{\theta}, \hat{\varphi}$ since they vary with position!

for example, $\hat{r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$

one could compute $\nabla^2\hat{r}$ by applying ∇^2 in spherical coords to each piece and summing up. Get a mess!

If work in Coulomb gauge, with $\vec{\nabla}\cdot\vec{A} = 0$, then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \boxed{-\nabla^2\vec{A} = \frac{4\pi}{c}\vec{j}} \quad \text{Poisson's equation!}$$

Many of the same methods used to solve for electrostatic ϕ can therefore be applied to solve for magnetostatic $\vec{A}$. But vector nature of eqn makes for complications!

For simple geometries, one can do the Coulomb-like integral

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$ three equations for A_x, A_y, A_z !

for localized current sources $j(r) \to 0$ as $r \to \infty$

<u>Multipole expansion</u> - magnetic dipole moment

For a general treatment, analogous to how we did multipole expansion for electrostatics, one can use <u>vector spherical harmonics</u> - see Jackson Chpt 9.

Here we do a more straight forward approach, but only up to magnetic dipole term.

For $r \gg r'$ approx

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{(r^2 - 2\vec{r}\cdot\vec{r}' + r'^2)^{1/2}} = \frac{1}{r}\,\frac{1}{\left[1 - \frac{2\vec{r}\cdot\vec{r}'}{r^2} + \left(\frac{r'}{r}\right)^2\right]^{1/2}}$$

do Taylor series to 1st order in $\left(\frac{r'}{r}\right)$ to get

$$\frac{1}{|\vec{r}-\vec{r}'|} \simeq \frac{1}{r}\left\{1 + \frac{\vec{r}\cdot\vec{r}'}{r^2} + \cdots\right\} = \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \cdots$$

$$\vec{A}(\vec{r}) = \overset{①}{\int \frac{d^3r'}{c} \frac{\vec{j}(\vec{r}')}{r}} + \overset{②}{\int \frac{d^3r'}{c} \vec{j}(\vec{r}') \frac{(\hat{r}\cdot\vec{r}')}{r^3}} + \cdots$$

Consider term ①

$$\int d^3r \, \vec{j}(\vec{r}) \qquad \overset{\displaystyle\int d^3r (\vec{j}\cdot\vec{\nabla})\vec{r}}{\shortparallel} \qquad \frac{\partial r_i}{\partial r_j} = \delta_{ij}$$

write $\int d^3r \, j_i(r) = \sum_{j=1}^{3} \int d^3r \, j_j \frac{\partial r_i}{\partial r_j}$ integrate by parts

for i^{th} component

$$= \sum_j \left\{ \oint_S da \, j_j \, r_i - \int d^3r \frac{\partial j_j}{\partial r_j} r_i \right\}$$

vanishes as $S \to \infty$ if $\vec{j}$ sufficiently localized ie $\vec{j}(\vec{r}) \to 0$ sufficiently fast as $r \to \infty$

vanishes in magnetostatics where $\vec{\nabla}\cdot\vec{j} = 0$

So $\int d^3r \, \vec{j}(\vec{r}) = 0$ in magnetostatics

monopole term vanishes

term (2)

$$\int d^3r \; \vec{j}(\vec{r})\,\vec{r} \qquad \text{tensor} \qquad \frac{\partial r_i}{\partial r_k} = \delta_{ik}$$

Consider $\int d^3r \; j_i r_j = \sum_k \int d^3r \; j_k r_j \frac{\partial r_i}{\partial r_k}$ integrate by parts
ij^{th} element of tensor

$$= \sum_k \left\{ \oint_S da \; j_k r_j r_i - \int d^3r \frac{\partial}{\partial r_k}(j_k r_j)\, r_i \right\}$$

vanishes as $S \to \infty$ if $\vec{j}$ sufficiently localized

$$= -\sum_k \int d^3r \left(\frac{\partial j_k}{\partial r_k} r_j r_i + j_k \frac{\partial r_j}{\partial r_k} r_i \right)$$

vanishes as $\vec{\nabla}\cdot\vec{j} = 0$ in magnetostatics $\qquad = \delta_{jk}$

$$= -\int d^3r \; j_j r_i$$

So $\int d^3r \; j_i r_j = -\int d^3r \; j_j r_i$

$$= \frac{1}{2}\int d^3r \left(j_i r_j - j_j r_i \right)$$

Going back to term (2) in expansion for $\vec{A}$

So

$$\int d^3r' \; j_i(\vec{r}')(\vec{r}\cdot\vec{r}') = \sum_{j=1}^{3} r_j \int d^3r' \; j_i(\vec{r}')\, r_j'$$

$$= \sum_j \frac{1}{2}\int d^3r' \left(j_i \, r_j r_j' - r_j j_j r_i' \right)$$

$$= \frac{1}{2}\int d^3r' \left(j_i (\vec{r}\cdot\vec{r}') - r_i' (\vec{r}\cdot\vec{j}) \right)$$

use triple product rule

$$\vec{r} \times (\vec{r}' \times \vec{J}) = \vec{r}'(\vec{r} \cdot \vec{J}) - \vec{J}(\vec{r} \cdot \vec{r}')$$

to rewrite as

$$\int d^3r' \vec{J}(\vec{r}, \vec{r}') = -\frac{1}{2} \vec{r} \times \left[\int d^3r' \, \vec{r}' \times \vec{J}(\vec{r}') \right]$$

<u>define</u> the magnetic dipole moment as

$$\boxed{\vec{m} \equiv \frac{1}{2c} \int d^3r' \, \vec{r}' \times \vec{J}(\vec{r}')}$$

then in the magnetic dipole approx (this is the lowest non-vanishing term)

$$\vec{A}_{dip}(\vec{r}) = -\frac{\vec{r} \times \vec{m}}{r^3} = \frac{\vec{m} \times \vec{r}}{r^3} = \frac{\vec{m} \times \hat{r}}{r^2}$$

What is the magnetic field in this approx?

$$\vec{B}_{dip} = \vec{\nabla} \times \vec{A}_{dip} = \vec{\nabla} \times \left(\vec{m} \times \frac{\vec{r}}{r^3} \right)$$

to do the double cross product, it is convenient to use the Levi-Civita symbol ε_{ijk} defined as

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{any two of the indices are equal} \\ +1 & ijk \text{ are an } \underline{\text{even}} \text{ permutation of } 123 \\ -1 & ijk \text{ are an } \underline{\text{odd}} \text{ permutation of } 123 \end{cases}$$

$i, j, k = 1, 2, 3$

In terms of Levi-Civita symbol $(\vec{A} \times \vec{B})_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} A_j B_k$

check by writing out $\vec{A} \times \vec{B}$ in terms of components

Summation conversion: when ever we have a pair of indices repeated, we sum over them, so

$$\varepsilon_{ijk} A_j B_k = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} A_j B_k$$

index j appears twice
index k appears twice

A <u>very</u> useful identity:

Kronecker deltas

$$\boxed{\varepsilon_{kij}\, \varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\,\delta_{jl}}$$

So we now put this to use!

$$\vec{B}_{dip} = \vec{\nabla} \times \left(\vec{m} \times \frac{\vec{r}}{r^3}\right)$$

i component

$$(B_{dip})_i = \varepsilon_{ijk}\, \partial_j\, \varepsilon_{klm}\, m_l \frac{r_m}{r^3} \qquad \text{where } \partial_j \equiv \frac{\partial}{\partial r_j}$$

$$= \varepsilon_{kij}\, \varepsilon_{klm}\, \partial_j \left(\frac{m_l r_m}{r^3}\right)$$

$\varepsilon_{ijk} = \varepsilon_{kij}$ as an even permutation takes one to the other

$$= (\delta_{il}\delta_{jm} - \delta_{im}\,\delta_{jl})\, \partial_j \left(m_l \frac{r_m}{r^3}\right)$$

$$= m_i\, \partial_j \left(\frac{r_j}{r^3}\right) - m_j\, \partial_j \left(\frac{r_i}{r^3}\right)$$

Now $\partial_j \left(\frac{r_j}{r^3}\right) = \vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3}\right) = \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\, \delta(\vec{r})$

$$\partial_j \left(\frac{r_i}{r^3}\right) = \frac{1}{r^3} \frac{\partial r_i}{\partial r_j} - \frac{3 r_i}{r^4} \frac{\partial r}{\partial r_j}$$

by product rule and $\frac{\partial r_i}{\partial r_j} = \delta_{ij}$

Now $\frac{\partial r}{\partial x} = \frac{\partial \sqrt{x^2+y^2+z^2}}{\partial x} = \frac{x}{r}$ so $\frac{\partial r}{\partial r_j} = \frac{r_j}{r}$

don't care about this term since we only want $\vec{B}$ far away from current where $\delta(\vec{r}) = 0$.

So

$$(B_{dip})_i = m_i \, 4\pi \delta(\vec{r}) - m_j \left[\frac{\delta_{ij}}{r^3} - \frac{3 r_i r_j}{r^5} \right]$$

$$= -\frac{m_i}{r^3} + \frac{3(\vec{m}\cdot\vec{r})\, r_i}{r^5}$$

$$= -\frac{m_i}{r^3} + \frac{3(\vec{m}\cdot\hat{r})\, \hat{r}_i}{r^3}$$

So

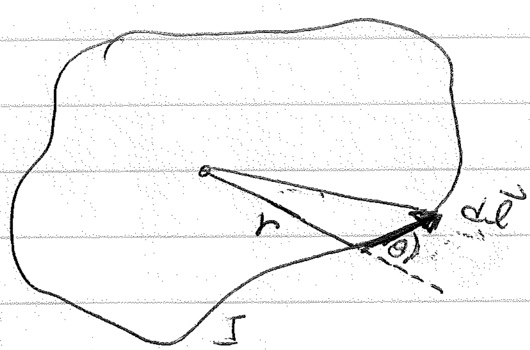
$$\boxed{\vec{B}_{dip} = \frac{3(\vec{m}\cdot\hat{r})\,\hat{r} - \vec{m}}{r^3}}$$

$$\boxed{\vec{B} = \frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3}}$$

same form as $\vec{E}$ from electric dipole $\vec{p}$

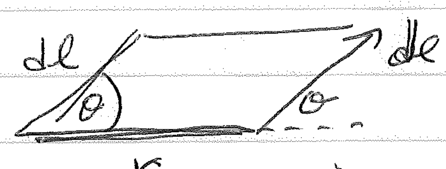
For a <u>current loop in a plane</u> (any shape loop provided it is flat)

$$\vec{m} = \frac{1}{2c}\int d^3r \; \vec{r}\times\vec{j} = \frac{1}{2c} I \oint \vec{r}\times d\vec{l}$$



area of triangle is $\frac{1}{2}\, r\, dl \sin\theta$

$= \frac{1}{2}\,|\vec{r}\times d\vec{l}\,|$



area of ~~rectangle~~ trapazoid is $r\, dl \sin\theta$

$$\Rightarrow \quad \vec{m} = \frac{1}{c} I(\text{area})\,\hat{m}$$

↑ area of loop

↖ outward normal (direction given by right hand rule with respect to direction of current) magnetic dipole moment $\vec{m}$ is independent of location of origin.

$$\vec{r}\,' = \vec{r} + \vec{d} \qquad \text{new coord}$$

$$\vec{m}\,' = \frac{1}{2c}\int d^3r' \left(\vec{r}\,' \times \vec{J}\right) = \frac{1}{2c}\int d^3r \, (\vec{r} + \vec{d}) \times \vec{J}$$

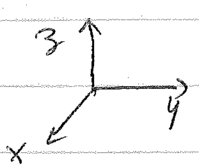
$$= \frac{1}{2c}\int d^3r \, \vec{r} \times \vec{J} + \frac{1}{2c}\vec{d} \times \left[\int d^3r \, \vec{J}\right]$$

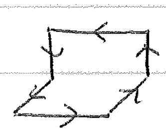
$$\vec{m}\,' = \vec{m} + 0 \quad \text{as} \quad \int d^3r \, \vec{J} = 0$$

---

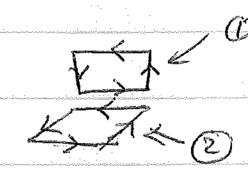
for planar loop $\vec{m} = \frac{I}{c} a \, \hat{n}$ where a = area
$\hat{n}$ = outward normal

Can also apply to get $\vec{m}$ for <u>piecewise planar loops</u>





= superposition of



$$\vec{m} = \vec{m}_1 + \vec{m}_2 \qquad \vec{m}_1 = \frac{I}{c} a_1 \hat{x}$$
$$\vec{m}_2 = \frac{I}{c} a_2 \hat{z}$$

$$\Rightarrow \quad \vec{m} = \frac{I}{c}\left(a_1 \hat{x} + a_2 \hat{z}\right)$$