Boundary value problems in magnetostatics
Scalar Magnetic Potential

Because of the vector character of the equation

$$-\nabla^2 \vec{A} = \frac{4\pi}{c}\vec{J}$$

and the fact that $\nabla^2 \vec{A}$ only has a convenient representation in Cartesian coordinates, many of the methods we used to solve the scalar $-\nabla^2\phi = 4\pi\rho$ dont work so well for magnetostatics.

However, in situations where the current $\vec{J}$ is confined to certain surfaces, we can make things much closer to the electrostatic case by using the trick of the scalar magnetic potential ϕ_M.

In regions where $\vec{J}=0$, ie not on the certain surfaces, we have $\vec{\nabla}\cdot\vec{B}=0$ and $\vec{\nabla}\times\vec{B}=0$. Since $\vec{\nabla}\times\vec{B}=0$ in these regions, we can define a scalar potential ϕ_M such that

$$\vec{B} = -\vec{\nabla}\phi_M$$

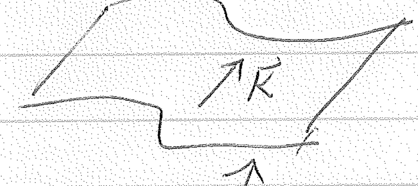
and then

$$\vec{\nabla}\cdot\vec{B} = -\nabla^2\phi_M = 0$$

We can solve for ϕ_M as in electrostatics, and match solutions by applying appropriate boundary conditions on the current carrying surfaces.

Boundary Conditions at sheet current

in magnetostatics $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j}$



surface current $\vec{K}(\vec{r})$ at pt $\vec{r}$ on surface S

surface S

Gaussian pillbox vol V

$$\int_V d^3r\, \vec{\nabla}\cdot\vec{B} = 0$$

side view

top + bottom area of pill box is da
width of pill box $\to 0$

$$\Rightarrow \int_V d^3r\, \vec{\nabla}\cdot\vec{B} = \oint_S da\, \hat{n}\cdot\vec{B} = da\left(\vec{B}_{above} - \vec{B}_{below}\right)\cdot\hat{n} = 0$$

<u>normal component of $\vec{B}$ is continuous</u> $(\vec{B}_{above} - \vec{B}_{below})\cdot\hat{n} = 0$

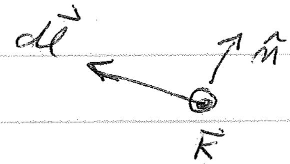
Amperian loop C bounding area S

$\vec{K}$ out of page

side view

$$\int_S da\, \hat{n}\cdot(\vec{\nabla}\times\vec{B}) = \oint_C \vec{B}\cdot d\vec{l} = \frac{4\pi}{c} I_{enclosed}$$

let width of loop $\to 0$, top + bottom sides $d\vec{l}$



$$\left(\vec{B}_{above} - \vec{B}_{below}\right)\cdot d\vec{l} = \frac{4\pi}{c}\overbrace{(\hat{n}\times d\vec{l})}^{\text{normal to Amperian loop } C}\cdot\vec{K}$$

$$= \frac{4\pi}{c}\left(\vec{K}\times\hat{n}\right)\cdot d\vec{l}$$

$\hat{n}$ is outward normal

<u>tangential component of $\vec{B}$ has discontinuous jump $\frac{4\pi}{c}\vec{K}\times\hat{n}$</u>

Combine both results into

$$\boxed{\vec{B}_{above} - \vec{B}_{below} = \frac{4\pi}{c} \vec{K} \times \hat{n}}$$

magnetic analog of $\vec{E}_{above} - \vec{E}_{below} = 4\pi\sigma \hat{n}$

In terms of magnetic ~~vector~~ scalar potential ϕ_M

$$-\vec{\nabla}\phi_{M\,above} + \vec{\nabla}\phi_{M\,below} = \frac{4\pi}{c} \vec{K} \times \hat{n}$$

Note: ϕ_M is a calculational tool only
it does not have any direct physical
significance as does the electrostatic ϕ.
Electrostatic ϕ is related to work done
moving a charge $W_{12} = q[\phi(r_2) - \phi(r_1)]$
nothing similar for ϕ_M.
(in fact ~~magnetostatic~~ magnetic forces do no work!

$$\vec{F} = q\,\vec{v} \times \vec{B}$$

$$\Rightarrow \vec{F} \cdot \vec{v} = \frac{dW}{dt} = 0 \;)$$

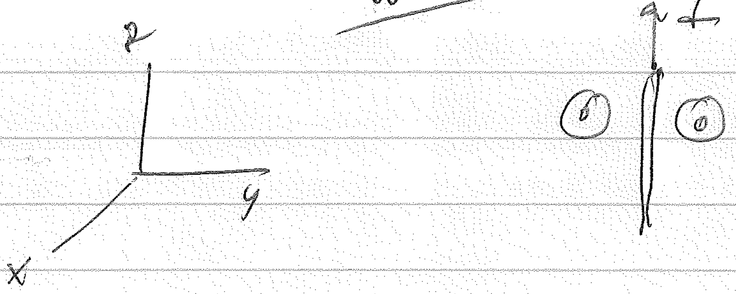
Note:

ϕ_M is not necessarily continuous at surface current
Cannot do similar to electrostatics and use

$$\phi_M(r_{above}) - \phi_M(r_{below}) = -\int_{r_{below}}^{r_{above}} \vec{B} \cdot d\vec{l}$$

since ϕ_M is not defined on the current sheet
itself, separating "above" from "below".

work on a current carrying wire



$\vec{I} = I\hat{z}$

$\vec{B} = B\hat{x}$

force per length on wire is

$$\vec{f} = \frac{\vec{I}}{c}\times\vec{B} = \frac{IB}{c}\hat{y}$$

work done on wire per time is $\vec{f}\cdot\vec{u}$ $\vec{u}$ is velocity of wire to right $\vec{u} = u\hat{x}$

$\frac{\text{Work}}{\text{time}}$ per length is $\frac{dW}{dt} = \frac{IBu}{c}$

Consider a charge moving in wire with $\vec{v} = \underbrace{v_z\hat{z}}_{\text{due to } I} + \underbrace{u\hat{y}}_{\text{due to motion of wire}}$

Force on charge is

$$\vec{F} = q\frac{\vec{v}\times\vec{B}}{c} = \frac{q v_z B}{c}\hat{y} - \frac{q}{c}uB\hat{z}$$

If current I ~~does not~~ stays constant, there must be increase in voltage down length of wire δV to create counter force along z

$$q\left(\frac{\delta V}{L}\right) = \frac{q}{c}uB \Rightarrow \delta V = \frac{uB}{c}L$$

$\frac{\text{work}}{\text{time}}$ done in the EV is $\delta V I = \frac{IuB}{c}L$

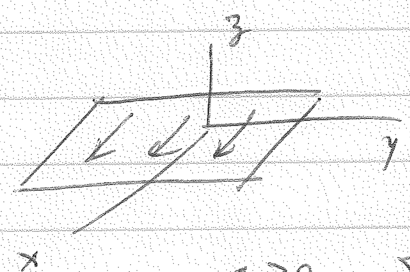
$$\frac{\text{work}}{\text{time length}} = \frac{IuB}{c}$$

battery does the work!

example

Flat infinite plane at $z=0$ with surface current

$$\vec{K} = K\hat{x}$$



$$z \geq 0, \quad \nabla^2\Phi_M^> = 0 \Rightarrow \Phi_M^> = a^> - b_x^> x - b_y^> y - b_z^> z$$
$$z < 0, \quad \nabla^2\Phi_M^< = 0 \Rightarrow \Phi_M^< = a^< - b_x^< x - b_y^< y - b_z^< z$$

$$z > 0, \quad \vec{B}^> = -\vec{\nabla}\Phi_M^> = b_x^>\hat{x} + b_y^>\hat{y} + b_z^>\hat{z}$$
$$z < 0, \quad \vec{B}^< = -\vec{\nabla}\Phi_M^< = b_x^<\hat{x} + b_y^<\hat{y} + b_z^<\hat{z}$$

at $z=0$ $\quad \vec{B}^> - \vec{B}^< = (b_x^> - b_x^<)\hat{x} + (b_y^> - b_y^<)\hat{y} + (b_z^> - b_z^<)\hat{z}$

$$= \frac{4\pi}{c}\vec{K}\times\hat{n} = \frac{4\pi}{c}K(\hat{x}\times\hat{z}) = -\frac{4\pi}{c}K\hat{y}$$

$$\Rightarrow \quad b_x^> = b_x^< \equiv b_{x0}, \quad b_z^> = b_z^< \equiv b_{z0}, \quad b_y^> - b_y^< = -\frac{4\pi K}{c}$$

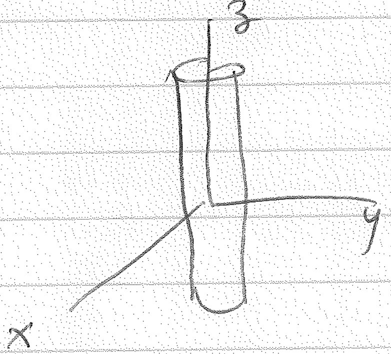
define
$$\left.\begin{array}{l} b_y^> = b_{y0} + \delta b_y \\ b_y^< = b_{y0} - \delta b_y \end{array}\right\} \quad \delta b_y = -\frac{2\pi K}{c}$$

$$\Rightarrow \quad \vec{B}^> = \vec{B}_0 - \frac{2\pi K}{c}\hat{y} \qquad \vec{B}_0 = b_{x0}\hat{x} + b_{y0}\hat{y} + b_{z0}\hat{z}$$
$$\vec{B}^< = \vec{B}_0 + \frac{2\pi}{c}K\hat{y}$$

if $\vec{K}$ is the only source of magnetic field then $\vec{B}_0 = 0$

$$\vec{B} = \begin{cases} -\frac{2\pi}{c}K\hat{y} & z > 0 \\ \frac{2\pi}{c}K\hat{y} & z < 0 \end{cases}$$

example current carrying infinite cylinder radius R



(i) $\vec{K} = K\hat{z}$ wire with surface current
(ii) $\vec{K} = K\hat{\varphi}$ solenoid

(i) $\vec{K} = K\hat{z}$ $\quad 2\pi R K = I$ total current

"guess" + show it is correct

$r>R$ $\quad \boxed{\Phi_M = -\frac{4\pi R K}{c}\varphi}$ $\quad$ magnetic scaler potential $\quad \nabla^2\Phi_M = 0$

$r<R$ $\quad \Phi_M = 0$

$r>R$ $\quad \vec{B} = -\vec{\nabla}\Phi_M = -\frac{1}{r}\frac{\partial\Phi_M}{\partial\varphi}\hat{\varphi} = \frac{4\pi R K}{c r}\hat{\varphi} = \boxed{\frac{2I}{cr}\hat{\varphi}}$ $\quad$ familiar result from Ampere

$r<R$ $\quad \vec{B} = 0$

$\vec{B}_{above} - \vec{B}_{below} = \frac{2I}{cR}\hat{\varphi} = \frac{4\pi K}{c}\frac{R}{R}\hat{\varphi} = \frac{4\pi}{c}\vec{K}\times\hat{n}$

where $\hat{n} = \hat{r}$

as $\hat{z}\times\hat{r} = \hat{\varphi}$

Note: $\Phi_M = -\frac{4\pi R K}{c}\varphi$ is not single valued!
would not have found this using expansion of separation of coords in polar coords
Φ_M does not need to be single valued since it has no physical significance. Only $\vec{B} = -\vec{\nabla}\Phi_M$ is physical

(ii) $\vec{K} = K\hat{\varphi}$

$r>R$ $\quad \Phi_M = -B_1 z$
$r<R$ $\quad \Phi_M = -B_2 z$ $\quad \Big\}\ \nabla^2\Phi_M = 0$

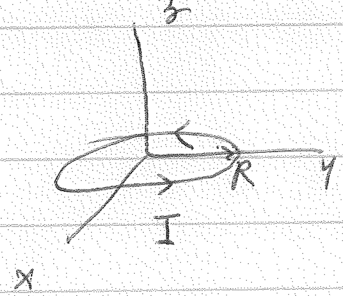
$r>R$ $\quad \vec{B} = -\vec{\nabla}\Phi_M = B_1\hat{z}$
$r<R$ $\quad \vec{B} = -\vec{\nabla}\Phi_M = B_2\hat{z}$

$$\vec{B}_{above} - \vec{B}_{below} = (B_1 - B_2)\,\hat{z} = \frac{4\pi}{c}\vec{K} \times \hat{n}$$
$$= \frac{4\pi}{c}K\,(\hat{\varphi} \times \hat{r})$$
$$= -\frac{4\pi}{c}K\,\hat{z}$$

If current in solenoid is only source of $\vec{B}$ Then expect $B_1 = 0$

$$\Rightarrow \boxed{\vec{B}_2 = \frac{4\pi}{c}K\,\hat{z}}$$ familiar result

example circular current loop in xy plane radius R



for $r > R$, $\vec{\nabla} \times \vec{B} = 0 \Rightarrow \vec{B} = -\vec{\nabla}\phi_M$

where $\nabla^2 \phi_M = 0$.

Try Legendre polynomial expansion for ϕ_M

$$\phi_M = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

(A_ℓ terms vanish as want $B \to 0$ as $r \to \infty$)

$$\vec{B} = -\vec{\nabla}\phi_M = -\frac{\partial \phi_M}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial \phi_M}{\partial \theta}\hat{\theta}$$

$$= \sum_\ell \left[\frac{(\ell+1)B_\ell}{r^{\ell+2}} P_\ell(\cos\theta)\,\hat{r} - \frac{B_\ell}{r^{\ell+2}} \frac{\partial P_\ell(\cos\theta)}{\partial \theta}\,\hat{\theta} \right]$$

write $\frac{\partial P_\ell}{\partial \theta} = \frac{\partial P_\ell}{\partial x}\frac{\partial x}{\partial \theta} = -\frac{\partial P_\ell}{\partial x}\sin\theta \equiv -P_\ell' \sin\theta$ $\quad x = \cos\theta$

$$\vec{B} = \sum_\ell \left[\frac{(\ell+1)B_\ell}{r^{\ell+2}} P_\ell(\cos\theta)\,\hat{r} + \frac{B_\ell}{r^{\ell+2}} \sin\theta\, P_\ell'(\cos\theta)\,\hat{\theta} \right]$$

To determine the B_ℓ we compare with exact solution along $\hat{z}$ axis

$$\vec{B}(z\hat{z}) = \sum_\ell \frac{(\ell+1)B_\ell}{r^{\ell+2}}\hat{r} = \sum_\ell \frac{(\ell+1)B_\ell}{z^{\ell+2}}\hat{z}$$

since $P_\ell(1) = 1$, $\sin(0) = 0$ and $P_\ell'(1)$ finite, $\hat{r} = \hat{z}$ when $\theta = 0$

exact solution on $\hat{z}$ axis:

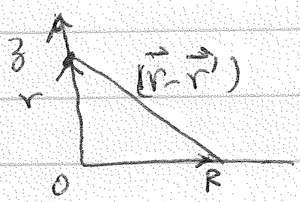
$$\vec{A} = \int \frac{d^3r'}{c} \frac{\vec{j}(r')}{|\vec{r}-\vec{r}'|} \Rightarrow \vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A} = \int \frac{d^3r'}{c}\, \vec{\nabla}\times \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\vec{B} = -\int \frac{d^3r'}{c}\, \vec{j}(\vec{r}') \times \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$$\vec{B} = \int \frac{d^3r'}{c}\, \vec{j}(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$ Biot-Savart Law for magnetostatics

For our loop $\quad \vec{B}(\vec{r}) = \frac{I}{c}\oint d\vec{l}' \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$

polar radial vector



$$\vec{B}(z) = \int_0^{2\pi} d\phi \frac{R}{c}\, I\hat{\phi} \times \frac{[z\hat{z} - R\hat{r}\ ]}{(z^2+R^2)^{3/2}}$$

$d\vec{l}' = d\phi R\hat{\phi}$

$$= \int_0^{2\pi} \frac{d\phi}{c} \frac{R(IR)\hat{z}}{(z^2+R^2)^{3/2}}$$

$\hat{\phi}\times\hat{z}$ term integrates to zero

$\hat{\phi}\times\hat{z} = \hat{r}$

$\hat{\phi}\times\hat{r} = -\hat{z}$

$$\boxed{\vec{B}(z) = \frac{2\pi R^2 I\, \hat{z}}{c(z^2+R^2)^{3/2}}}$$

to match Legendre polynomial expansion, do Taylor series expansion of above

$$\vec{B}(z) = \frac{2\pi R^2 I\, \hat{z}}{c z^3} \frac{1}{\left(1+\left(\frac{R}{z}\right)^2\right)^{3/2}} = \frac{2\pi R^2 I\, \hat{z}}{z^3}\left\{1 - \frac{3}{2}\left(\frac{R}{z}\right)^2 + \cdots\right\}$$

$$= \frac{2\pi R^2 I\, \hat{z}}{c}\left\{\frac{1}{z^3} - \frac{3}{2}\frac{R^2}{z^5} + \cdots\right\}$$

$$= \left\{\frac{B_0}{z^2} + \frac{2B_1}{z^3} + \frac{3B_2}{z^4} + \frac{4B_3}{z^5} + \cdots\right\}\hat{z}$$

$$\Rightarrow B_0 = 0, \quad B_1 = \frac{\pi R^2 I}{c}, \quad B_2 = 0, \quad B_3 = -\frac{3}{4c}\pi R^2 I R^2$$

So to order $l = 3$

$$\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c}\left\{ \frac{2P_1(\cos\theta)\hat{r} + \sin\theta P_1'(\cos\theta)\hat{\theta}}{r^3} - \left[\frac{3R^2 P_3(\cos\theta)\hat{r} + \frac{3}{4}R^2 \sin\theta P_3'(\cos\theta)\hat{\theta}}{r^5}\right] + \cdots \right\}$$

$$P_1(x) = x \Rightarrow P_1'(x) = 1$$

$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x) \Rightarrow P_3'(x) = \tfrac{1}{2}(15x^2 - 3)$$

$$\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c}\left\{ \frac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3} - \left[\frac{\frac{3}{2}R^2(5\cos^3\theta - 3\cos\theta)\hat{r} + \frac{3}{8}R^2\sin\theta(15\cos^2\theta - 3)\hat{\theta}}{r^5}\right] + \cdots \right\}$$

$\frac{\pi R^2 I}{c} = m$ is the magnetic dipole moment of the loop

We see that the 1st term is just the magnetic dipole approx. The 2nd term is the magnetic octopole term. Could easily get higher order terms by this method.

Compare our result above to Jackson (5-40)