Linear Materials

Macroscopic Maxwell Equations

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \qquad \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$

where ρ and $\vec{j}$ are macroscopic charge + current densities and

$$\vec{D} = \vec{E} + 4\pi\vec{P} \qquad \vec{P} \text{ is polarization density}$$
$$\vec{H} = \vec{B} - 4\pi\vec{M} \qquad \vec{M} \text{ is magnetization density}$$

To close these equations, we will in general need to know how $\vec{P}$ and $\vec{M}$ are related to the $\vec{E}$ and $\vec{B}$ in the material.

In some materials, there can be a finite $\vec{P}$ or $\vec{M}$ even if $\vec{E}$ and $\vec{B}$ are zero:

<u>Ferromagnet</u>: $\vec{M}$ can be non zero even if $\vec{B} = 0$
<u>Ferroelectric</u>: $\vec{P}$ can be non zero even if $\vec{E} = 0$

But more common are <u>linear</u> materials in which, for <u>small</u> $\vec{E}$ and $\vec{B}$, one has $\vec{P} \propto \vec{E}$ and $\vec{M} \propto \vec{B}$.

linear dielectric

$\vec{P} = \chi_e \vec{E}$ $\quad$ χ_e is "electric susceptibility"

$\chi_e > 0$ for statics

$\vec{D} = \vec{E} + 4\pi \vec{P} = (1 + 4\pi \chi_e) \vec{E}$

$\vec{D} = \epsilon \vec{E}$ with $\epsilon = 1 + 4\pi \chi_e$

ϵ is the dielectric constant

linear magnetic materials

$\vec{M} = \chi_m \vec{H}$ $\quad$ χ_m is "magnetic susceptibility"

$\chi_m > 0 \Rightarrow$ paramagnetic

$\chi_m < 0 \Rightarrow$ diamagnetic

$\vec{H} = \vec{B} - 4\pi \vec{M} = \vec{B} - 4\pi \chi_m \vec{H}$

$\vec{B} = (1 + 4\pi \chi_m) \vec{H}$

$\vec{B} = \mu \vec{H}$ with $\mu = 1 + 4\pi \chi_m$

μ is magnetic permeability

For statics, $\chi_e > 0$ and χ_m (or alternatively ϵ and μ) are constants depending on the material.

When we consider dynamics we will see that ϵ becomes a function of frequency.

Clausius - Mossotti equation

Electric susceptibility & atomic polarizability

If a field $\vec{E}_{loc}$ is applied to an atom, it gets polarized

$$\vec{p} = \alpha \vec{E}_{loc}$$

"local field" - field the atom sees

atomic dipole moment

atomic polarizability

α is what one calculates from a microscopic theory

If $\vec{E}_{loc} = \vec{E}$ the average field in the material then electric susceptibility given by

$$\vec{P} = n\vec{p} = n\alpha \vec{E}_{loc} = n\alpha\vec{E} = \chi_e \vec{E}$$

$\Rightarrow \chi_e = n\alpha$ where n = density of atoms

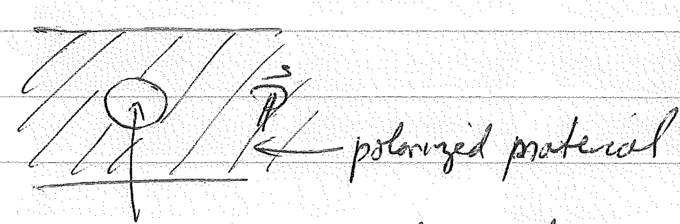
But a more careful consideration shows $\vec{E}_{loc} \neq \vec{E}$. The average field $\vec{E}$ includes the electric field created by the polarized atom itself. $\vec{E}_{loc}$, the local field the atom sees, should exclude its own self field.

$$\vec{E} = \vec{E}_{loc} + \vec{E}_{atom}$$

average field

average field excluding atom

average field of the atom



cut out sphere whose volume is $1/n$ the volume per atom

$\vec{E}_{loc}$ is field excluding the field of the polarized sphere of volume $1/n$.

$\vec{E}_{atom}$ is field of the polarized sphere

$$\vec{E}_{atom} = -\frac{4\pi \vec{P}}{3} = -\frac{4\pi}{3} n \vec{p}$$

$$\vec{E}_{loc} = \vec{E} - \vec{E}_{atom} = \vec{E} + \frac{4\pi}{3}\vec{P} = \vec{E} + \frac{4\pi}{3} n \vec{p}$$

$$\vec{p} = \alpha \vec{E}_{loc} = \alpha\left(\vec{E} + \frac{4\pi}{3} n \vec{p}\right) = \alpha \vec{E} + \frac{4\pi}{3} n\alpha\, \vec{p}$$

$$\vec{p} = \frac{\alpha}{1 - \frac{4\pi}{3} n\alpha} \vec{E}$$

$$\vec{P} = n\vec{p} = \frac{\alpha n}{1 - \frac{4\pi}{3} n\alpha} \vec{E} = \chi_e \vec{E}$$

$$\boxed{\chi_e = \frac{n\alpha}{1 - \frac{4\pi}{3} n\alpha}}$$

or solve for α in terms of ϵ

$$\chi_e = \frac{n\alpha}{1 - \frac{4\pi}{3} n\alpha} \Rightarrow \chi_e - \frac{4\pi}{3} n\alpha\chi_e = \alpha n$$

$$\Rightarrow \alpha = \frac{\chi_e}{n(1 + \frac{4\pi}{3}\chi_e)}$$

$$\epsilon = 1 + 4\pi\chi_e \Rightarrow \alpha = \frac{\epsilon - 1}{4\pi n} \frac{1}{\left(1 + \frac{\epsilon - 1}{3}\right)}$$

relates atomic polarizability to measured dielectric constant

$$\boxed{\alpha = \frac{3}{4\pi n}\left(\frac{\epsilon - 1}{\epsilon + 2}\right)}$$

Clausius Mossotti or Lorentz-Lorenz equation

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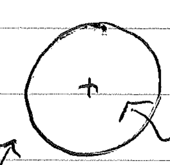
simple model for α



uniform $\rho = \frac{-q}{\frac{4}{3}\pi a^3}$

atomic radius a

field inside is $E(r) = \frac{4\pi\rho}{3} r \hat{r}$

In external field E_0, electron cloud will be displaced from core by distance d. We can determine d by making a balance of forces: $\Rightarrow qE_0 = q\frac{4\pi\rho d}{3}$

$$p = qd = \frac{3}{4\pi\rho} qE_0 = \frac{3\left(\frac{4}{3}\pi a^3\right)}{4\pi q} qE_0$$

$$= a^3 E_0 \Rightarrow \alpha = a^3$$

$$\chi_e = \frac{na^3}{1 - \frac{4\pi}{3} na^3}$$

if $f = n\frac{4}{3}\pi a^3$ fraction of vol that is occupied by atoms

$$\boxed{\chi_e = \frac{1}{4\pi}\frac{3f}{1-f}}$$

Linear dielectrics

bound charge is proportional to free charge

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot(\chi_e\vec{E}) = -\vec{\nabla}\cdot\left(\frac{\chi_e}{\epsilon}\vec{D}\right)$$

if χ_e (and hence ϵ) is spatially constant, then

$$\rho_b = -\frac{\chi_e}{\epsilon}\vec{\nabla}\cdot\vec{D} = -\frac{\chi_e}{\epsilon}4\pi\rho$$

$$\boxed{\rho_b = -\frac{4\pi\chi_e}{1+4\pi\chi_e}\rho}$$

when free charge $\rho = 0$, then $\rho_b = 0$

$$\rho_{total} = \rho + \rho_b = \rho\left[1 - \frac{4\pi\chi_e}{1+4\pi\chi_e}\right] = \frac{\rho}{1+4\pi\chi_e} = \boxed{\frac{\rho}{\epsilon} = \rho_{total}}$$

bound charge "screens" the free charge so the total charge is reduced compared to the free charge.

For linear dielectrics

Statics

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho$$
$$\vec{\nabla}\times\vec{E} = 0$$

$$\vec{D} = \epsilon\vec{E} \quad\Rightarrow\quad \vec{\nabla}\cdot(\epsilon\vec{E}) = 4\pi\rho$$

if ϵ is constant in space then $\epsilon\,\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

$$\left.\begin{aligned}\vec{\nabla}\cdot\vec{E} &= 4\pi\rho/\epsilon = 4\pi\rho_{tot}\\ \vec{\nabla}\times\vec{E} &= 0\end{aligned}\right\}$$ look just like ordinary electrostatics but with $\rho \to \rho/\epsilon$

Alternatively, could write $\vec{E} = \vec{D}/\epsilon$

$$\Rightarrow \vec{\nabla}\times(\vec{D}/\epsilon) = 0$$
$$\Rightarrow \vec{\nabla}\times\vec{D} = 0 \quad \text{when } \epsilon \text{ constant in space}$$

$$\left.\begin{aligned}\vec{\nabla}\cdot\vec{D} &= 4\pi\rho\\ \vec{\nabla}\times\vec{D} &= 0\end{aligned}\right\}$$ looks just like ordinary electrostatics, but with $\vec{E}\to\vec{D}$

Complication arises at interface between dielectrics (or between dielectric and vacuum). At interface, ϵ is not constant $\Rightarrow \vec{\nabla}\times\vec{D} \neq 0$.

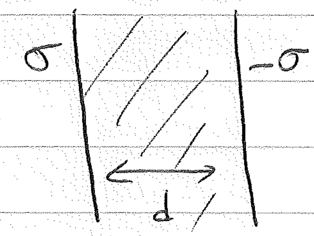
What we can do is to solve for $\vec{E}$ or $\vec{D}$ inside each dielectric separately, and then use the boundary conditions

$$\hat{n}\cdot(\vec{D}^{above} - \vec{D}^{below}) = 4\pi\sigma$$
$$\hat{t}\cdot(\vec{E}^{above} - \vec{E}^{below}) = 0$$

to match solutions across the interfaces.

A similar story holds for linear magnetic materials

Simple example: parallel plate capacitor filled with a dielectric



σ free charge

What is E between plates?

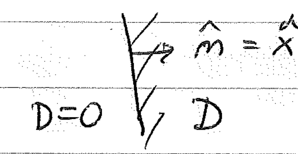
we know $\vec{E} = \vec{D} = 0$ outside plates

Between plates $\vec{\nabla} \cdot \vec{D} = 0$ as $\rho = 0$

$\vec{D} = D(x)\,\hat{x} \Rightarrow \frac{\partial D}{\partial x} = 0 \Rightarrow D$ is constant

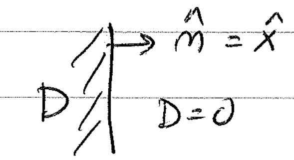
Boundary conditions:

left side plate



$$\hat{x} \cdot (\vec{D}^{\text{above}} - \vec{D}^{\text{below}}) = D = 4\pi\sigma$$

right side plate



$$\hat{x} \cdot (\vec{D}^{\text{above}} - \vec{D}^{\text{below}}) = -D = 4\pi(-\sigma)$$

$D = 4\pi\sigma$ as before

$$\Rightarrow \vec{D} = 4\pi\sigma\,\hat{x}$$

$$\boxed{\vec{E} = \frac{\vec{D}}{\epsilon} = \frac{4\pi\sigma}{\epsilon}\,\hat{x}}$$

electric field reduced by factor $1/\epsilon$ as compared to capacitor with vacuum between plates

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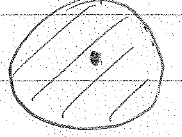
see Jackson section 4.4 for more interesting examples
– dielectric sphere in uniform applied $\vec{E}$

see Jackson section (5.11) 5.12 for an interesting magnetic b.c. problem
– spherical permeable shell in uniform applied $\vec{B}$

point charge within a dielectric sphere



pt charge q at center of dielectric sphere of radius R, dielectric const ϵ

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \quad = \quad \oint_S da\, \hat{n}\cdot\vec{D} = 4\pi Q_{encl}$$

From symmetry $\vec{D}(\vec{r}) = D(r)\,\hat{r}$

$$\oint_S da\, \hat{n}\cdot\vec{D} = 4\pi r^2 D(r) = 4\pi q$$

sphere of radius r $\nearrow$ S

$$\vec{D} = \frac{q}{r^2}\,\hat{r} \qquad \text{all } r$$

$$\Rightarrow \quad \vec{E}(\vec{r}) = \begin{cases} \dfrac{q}{\epsilon r^2}\,\hat{r} & r < R \\[2ex] \dfrac{q}{r^2}\,\hat{r} & r > R \end{cases}$$

can check that tangential component of $\vec{E}$ is continuous and normal component of $\vec{D}$ is continuous as there is no free σ at surface of dielectric.

normal component of $\vec{E}$ jumps by

$$\hat{n}\cdot(\vec{E}^{above} - \vec{E}^{below}) = \frac{q}{R^2} - \frac{q}{\epsilon R^2} = \frac{q}{R^2}\left(1 - \frac{1}{\epsilon}\right) = \frac{q}{R^2}\left(\frac{\epsilon - 1}{\epsilon}\right)$$

$$= \frac{q}{R^2}\left(\frac{4\pi\chi_e}{1+4\pi\chi_e}\right) = 4\pi\sigma_{total} = 4\pi\sigma_b$$

$$\Rightarrow \quad \sigma_b = \frac{q}{4\pi R^2}\left(\frac{4\pi\chi_e}{1+4\pi\chi_e}\right) = \frac{q\,\chi_e}{R^2\epsilon}$$

We can check this directly

$$\vec{\mathbb{P}} = \chi_e \vec{E} = \frac{\chi_e}{\epsilon} \frac{q}{r^2} \hat{r}$$

$$\rho_b = -\vec{\nabla} \cdot \vec{\mathbb{P}} = -\frac{\chi_e}{\epsilon} q \, 4\pi \delta(\vec{r})$$

bound charge at origin $q_b = -\frac{\chi_e}{\epsilon} 4\pi q$

total charge at origin is $q + q_b = q\left(1 - \frac{4\pi \chi_e}{\epsilon}\right)$

$\epsilon = 1 + 4\pi \chi_e$ $\quad = q\left(\frac{\epsilon - 4\pi \chi_e}{\epsilon}\right) = \frac{q}{\epsilon}$ screened charge

at surface,

$$\sigma_b = \hat{n} \cdot \vec{\mathbb{P}} = \frac{\chi_e}{\epsilon} \frac{q}{R^2}$$

agrees with what we get from jump in $\hat{n} \cdot \vec{E}$.

<u>Note:</u> inside the dielectric the $\vec{E}$ field is that of the screened point charge $\frac{q}{\epsilon}$

outside the dielectric $\vec{E}$ is just that of the free charge q. There is no evidence in $\vec{E}_{out}$ that the dielectric even exists!