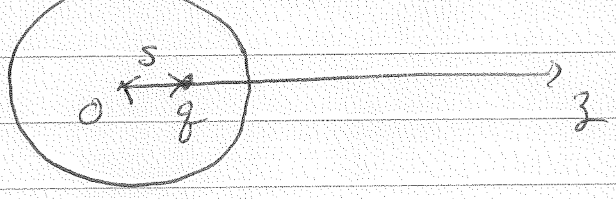
Now consider same problem but q is off center



what is $\vec{E}$ inside + outside

<u>inside</u> $\vec{\nabla}\cdot\vec{D} = 4\pi\rho$ where $\rho = q\,\delta(\vec{r}-s\hat{z})$

$\vec{D} = \epsilon\vec{E} \Rightarrow \vec{\nabla}\cdot\vec{E} = 4\pi\rho/\epsilon$

$\vec{E} = -\vec{\nabla}\phi \Rightarrow \nabla^2\phi = -\dfrac{4\pi\rho}{\epsilon} = -4\pi\dfrac{q}{\epsilon}\,\delta(\vec{r}-s\hat{z})$

solution for ϕ will be of the form

$$\phi(\vec{r}) = \frac{q}{\epsilon|\vec{r}-s\hat{z}|} + F(\vec{r})$$

where 1st term is due to the point charge q/ϵ and 2nd term satisfies $\nabla^2 F = 0$ and will be chosen to get the correct behavior at the boundary of the dielectric

Since there is azimuthal symmetry about $\hat{z}$ we can write

$$F(\vec{r}) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta)$$

there are no $\dfrac{b_l}{r^{l+1}}$ terms since F should not diverge at the origin

So inside, $r < R$

$$\phi^{in}(\vec{r}) = \frac{q}{\epsilon|\vec{r}-s\hat{z}|} + \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell(\cos\theta)$$

From our discussion of electric multipole expansion, we know we can write for $r > s$,

$$\frac{1}{|\vec{r}-s\hat{z}|} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{s}{r}\right)^\ell P_\ell(\cos\theta)$$

So for $\underline{\underline{r>s}}$ (not true for $r<s$!)

$$\phi^{in}(\vec{r}) = \sum_{\ell=0}^{\infty}\left(\frac{q}{\epsilon r}\left(\frac{s}{r}\right)^\ell + a_\ell r^\ell\right)P_\ell(\cos\theta)$$

Outside the sphere there is no charge, so $\vec{\nabla}\cdot\vec{E}=0$ or $\nabla^2\phi = 0$

$$\Rightarrow \phi^{out}(\vec{r}) = \sum_{\ell=0}^{\infty}\frac{b_\ell}{r^{\ell+1}}P_\ell(\cos\theta)$$

there are no $a_\ell r^\ell$ terms since $\phi^{out}\to 0$ as $r\to\infty$

To determine the unknown a_ℓ and b_ℓ we use the boundary conditions at surface of dielectric at $r=R$

① Tangential component $\vec{E}$ is continuous

$$\vec{E} = -\frac{\partial \phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\theta} = E_r \hat{r} + E_\theta \hat{\theta}$$

$\Rightarrow$ E_θ is continuous at $r = R$

condition that E_θ is continuous is the same condition that ϕ is continuous (check this out for yourself if you are not sure)

as $\vec{E}^{above} - \vec{E}^{below} = 4\pi\sigma_{tot}$

$$\Rightarrow \phi^{in}(R,\theta) = \phi^{out}(R,\theta)$$

$$\frac{q}{\epsilon R}\left(\frac{s}{R}\right)^l + a_l R^l = \frac{b_l}{R^{l+1}}$$

$$\Rightarrow \boxed{b_l = \frac{q}{\epsilon} s^l + a_l R^{2l+1}}$$

normal component $\vec{D}$ is continuous (since free surface charge $\sigma = 0$)

$$\vec{D} = \epsilon \vec{E}$$

$$\Rightarrow \epsilon E_r^{in} = E_r^{out}$$

$$-\epsilon \left.\frac{\partial \phi^{in}}{\partial r}\right|_R = -\left.\frac{\partial \phi^{out}}{\partial r}\right|_R$$

$$\Rightarrow \frac{(l+1)q}{R^2}\left(\frac{s}{R}\right)^l - l\epsilon a_l R^{l-1} = \frac{(l+1)b_l}{R^{l+2}}$$

$$q s^\ell - \frac{\ell}{\ell+1} \epsilon a_\ell R^{2\ell+1} = b_\ell$$

substitute in b_ℓ from previous boundary condition

$$q s^\ell - \frac{\ell}{\ell+1} \epsilon a_\ell R^{2\ell+1} = \frac{q}{\epsilon} s^\ell + a_\ell R^{2\ell+1}$$

$$q s^\ell \left[1 - \frac{1}{\epsilon}\right] = a_\ell R^{2\ell+1} \left[1 + \frac{\ell}{\ell+1} \epsilon\right]$$

$$\boxed{a_\ell = \frac{q s^\ell}{R^{2\ell+1}} \frac{\left[1 - \frac{1}{\epsilon}\right]}{\left[1 + \left(\frac{\ell}{\ell+1}\right)\epsilon\right]}}$$

$$b_\ell = \frac{q}{\epsilon} s^\ell + a_\ell R^{2\ell+1}$$

$$= \frac{q}{\epsilon} s^\ell + q s^\ell \frac{\left[1 - \frac{1}{\epsilon}\right]}{\left[1 + \left(\frac{\ell}{\ell+1}\right)\epsilon\right]}$$

$$b_\ell = \frac{q s^\ell}{\epsilon} \left\{1 + \frac{\epsilon - 1}{1 + \left(\frac{\ell}{\ell+1}\right)\epsilon}\right]$$

$$= \frac{q s^\ell}{\epsilon} \left[\frac{\epsilon \left(1 + \frac{\ell}{\ell+1}\right)}{1 + \left(\frac{\ell}{\ell+1}\right)\epsilon}\right]$$

$$\boxed{b_\ell = q s^\ell \left[\frac{1 + \left(\frac{\ell}{\ell+1}\right)}{1 + \left(\frac{\ell}{\ell+1}\right)\epsilon}\right]}$$

check the result:

as $s \to 0$, should recover previous answer

for $s = 0$, $a_\ell = b_\ell = 0$ for all $\ell \neq 0$

$$a_0 = \frac{q}{R}\left[1 - \frac{1}{\epsilon}\right]$$

$$b_0 = q$$

so $\phi^{in}(\vec{r}) = \dfrac{q}{\epsilon r} + \dfrac{q}{R}\left[1 - \dfrac{1}{\epsilon}\right]$

$$\vec{E}^{in} = -\vec{\nabla}\phi^{in} = \frac{q}{\epsilon r^2}\hat{r} \quad \text{as before}$$

$$\phi^{out}(\vec{r}) = \frac{q}{r}$$

$$\vec{E}^{out} = -\vec{\nabla}\phi^{out} = \frac{q}{r^2}\hat{r} \quad \text{as before}$$

note: the constant that is the 2nd term in ϕ^{in} is just what is needed to make ϕ continuous at $r = R$

another check:

let $\epsilon \to \infty$ this models a conductor!

again one finds $a_\ell = b_\ell = 0$ for all $\ell \neq 0$

$$a_0 = \frac{q}{R}$$

$$b_0 = q$$

$$\phi^{in}(\vec{r}) = \sum_\ell \frac{q}{\epsilon r}\left(\frac{s}{r}\right)^\ell P_\ell + \frac{q}{R} \to \frac{q}{R} \quad \text{as } \epsilon \to \infty$$

$$\Rightarrow E^{in}(\vec{r}) = 0 \quad \text{as } \phi^{in} \text{ is a constant.}$$

$$\phi^{out}(\vec{r}) = \frac{q}{r} \Rightarrow \vec{E}^{out} = \frac{q}{r^2}\hat{r}$$

field outside is like point charge q at the origin, independent of where q is inside the sphere. This is the correct behavior of a conductor. The mobile charges in the conductor completely screen the q inside, and leave a uniform surface charge $\sigma_b = \frac{q}{4\pi R^2}$ on the surface.

Magnetostatics

Bar magnets - $\vec{j}=0$, $\vec{M}$ fixed and given
(not a linear material)

$$\vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{H}=\frac{4\pi}{c}\vec{j}=0$$

$\vec{\nabla}\times\vec{H}=0 \Rightarrow \vec{H}=-\vec{\nabla}\Phi_M$ magnetic scalar potential

$$\vec{B}=\vec{H}+4\pi\vec{M}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{B}=\vec{\nabla}\cdot(\vec{H}+4\pi\vec{M})=0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{H}=-\nabla^2\Phi_M=-4\pi\,\vec{\nabla}\cdot\vec{M}$$

$\nabla^2\Phi_M = 4\pi\,\vec{\nabla}\cdot\vec{M}$ Poisson's equation! same form as electrostatics

so $\rho_M \equiv -\vec{\nabla}\cdot\vec{M}$ looks like a magnetic "charge"

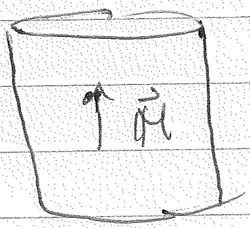
ρ_M is source for $\vec{H}$

Also at surfaces of material $\sigma_M=\hat{n}\cdot\vec{M}$ looks like surface charge (just like $\rho_b=-\vec{\nabla}\cdot\vec{P}$ and $\sigma_b=\hat{n}\cdot\vec{P}$)
for bound charge density and bound surface charge density

$$\vec{H}(\vec{r})=\int_V d^3r'\,\rho_M(\vec{r}\,')\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}+\oint_S da'\,\sigma_M(\vec{r}\,')\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

Field lines for $\vec{H}$ can start and end at sources and sinks given by ρ_M and σ_M

(field lines for $\vec{B}$ must still be continuous with no sources or sinks since we still have $\vec{\nabla}\cdot\vec{B}=0$)



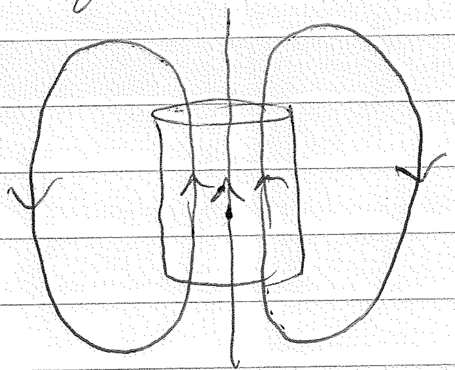
$\vec{M} = M\hat{z}$

bound currents $\vec{J}_b = c\,\vec{\nabla}\times\vec{M} = 0$

$\vec{K}_b = c\,\vec{M}\times\hat{n}$

$\vec{K}_b = \begin{cases} cM\hat{\phi} & \text{on side} \\ 0 & \text{on top + bottom} \end{cases}$

$\vec{K}_b$ is like solenoid current
field lines of $\vec{B}$ look like

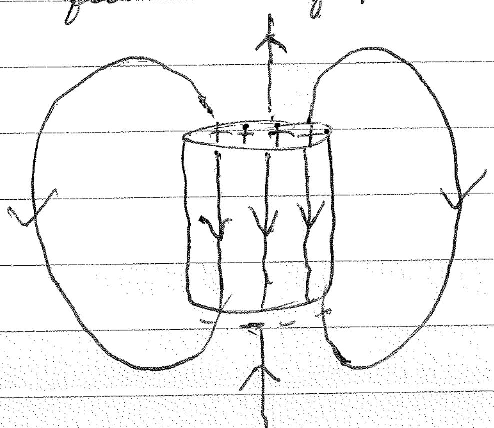


But $\vec{H}$ is determined as follows:

$\rho_M = -\vec{\nabla}\cdot\vec{M} = 0$

$\sigma_M = \vec{n}\cdot\vec{M} = \begin{cases} M & \text{on top} \\ -M & \text{on bottom} \end{cases}$

field lines of $\vec{H}$ look like parallel plate capacitor



field lines of $\vec{H}$ = field lines of $\vec{B}$ outside magnet, but they are very different inside the magnet!

Conservation of Energy

– leave macroscopic Maxwell eqns for present. $\vec{E}$, $\vec{B}$, ρ, $\vec{J}$ are now the exact microscopic quantities

Consider a collection of charged particles, described by charge density ρ and current density $\vec{J}$. The particles are contained in a volume V.

Define $\mathcal{E}_{mech}$ as total "mechanical" energy of the particles. $\mathcal{E}_{mech}$ = sum of particles kinetic energy plus potential energy of any non-electromagnetic forces.

The particles will exert forces on each other via their electromagnetic interactions, ie via the $\vec{E}$ and $\vec{B}$ fields that they create. Define W as the work done on the particles by all electromagnetic forces. Then, by the work energy theorem of mechanics:

$$\frac{d\mathcal{E}_{mech}}{dt} = \frac{dW}{dt}$$

For a single charge q_i, (at $\vec{r}_i$ with velocity $\vec{v}_i$)

$$\frac{dW}{dt} = \vec{F}_i \cdot \vec{v}_i$$

$$= q_i \vec{E}(\vec{r}_i) \cdot \vec{v}_i + q\underbrace{\left(\frac{\vec{v}_i}{c} \times \vec{B}\right) \cdot \vec{v}_i}_{= 0}$$

$$= q_i \vec{E}(r_i) \cdot \vec{v}_i$$

For the collection of charges, with

$$\vec{j}(\vec{r},t) = \sum_i q_i \vec{v}_i \, \delta(\vec{r}-\vec{r}_i(t))$$

the total rate of work done is

$$\frac{dW}{dt} = \sum_i q_i \vec{v}_i \cdot \vec{E}(\vec{r}_i) = \int_V d^3r \, \vec{j}\cdot\vec{E}$$

So

$$\frac{d\mathcal{E}_{mech}}{dt} = \int_V d^3r \, \vec{j}\cdot\vec{E}$$

or writing $u_{mech} = \mathcal{E}_{mech}/V$
then $\frac{\partial u_{mech}}{\partial t} = \vec{j}\cdot\vec{E}$

By maxwell equation $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$
we can write

$$\vec{j} = \frac{c}{4\pi}\left[\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right]$$

$$\int_V d^3r \, \vec{j}\cdot\vec{E} = \int_V d^3r \frac{c}{4\pi}\left[(\vec{\nabla}\times\vec{B})\cdot\vec{E} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\cdot\vec{E}\right]$$

use $\frac{\partial\vec{E}}{\partial t}\cdot\vec{E} = \frac{1}{2}\frac{\partial E^2}{\partial t}$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = (\vec{\nabla}\times\vec{E})\cdot\vec{B} - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$$

$$\Rightarrow \vec{E}\cdot(\vec{\nabla}\times\vec{B}) = (\vec{\nabla}\times\vec{E})\cdot\vec{B} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

then use $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$

So $$\vec{E}\cdot(\vec{\nabla}\times\vec{B}) = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\cdot\vec{B} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

$$= -\frac{1}{2c}\frac{\partial B^2}{\partial t} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$$