

For the collection of charges, with

$$\vec{j}(\vec{r}, t) = \sum_i q_i \vec{v}_i \delta(\vec{r} - \vec{r}_i(t))$$

the total rate of work done is

$$\frac{dW}{dt} = \sum_i q_i \vec{v}_i \cdot \vec{E}(\vec{r}_i) = \int_V d^3r \vec{j} \cdot \vec{E}$$

So

$$\frac{dE_{\text{mech}}}{dt} = \int_V d^3r \vec{j} \cdot \vec{E}$$

or writing  $u_{\text{mech}} = E_{\text{mech}}/V$   
then  $\frac{dE_{\text{mech}}}{dt} = \vec{j} \cdot \vec{E}$

By Maxwell equation  $\vec{\nabla} \times \vec{B} = 4\pi \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$   
we can write

$$\vec{j} = \frac{c}{4\pi} \left[ \vec{\nabla} \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right]$$

$$\int_V d^3r \vec{j} \cdot \vec{E} = \int_V d^3r \frac{c}{4\pi} \left[ (\vec{\nabla} \times \vec{B}) \cdot \vec{E} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \cdot \vec{E} \right]$$

use  $\frac{\partial \vec{E}}{\partial t} \cdot \vec{E} = \frac{1}{2} \frac{\partial E^2}{\partial t}$

$$\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = (\vec{\nabla} \times \vec{E}) \cdot \vec{B} - \vec{E} \cdot (\vec{\nabla} \times \vec{B})$$

$$\Rightarrow \vec{E} \cdot (\vec{\nabla} \times \vec{B}) = (\vec{\nabla} \times \vec{E}) \cdot \vec{B} - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

then use  $\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$

$$\text{So } \vec{E} \cdot (\vec{\nabla} \times \vec{B}) = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \cdot \vec{B} - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

$$= -\frac{1}{2c} \frac{\partial B^2}{\partial t} - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

Combine results to get

$$\int_V d^3r \vec{j} \cdot \vec{E} = -\frac{1}{4\pi} \int_V d^3r \left[ \frac{1}{2} \frac{\partial B^2}{\partial t} + \frac{1}{2} \frac{\partial E^2}{\partial t} + c \vec{\nabla} \cdot (\vec{E} \times \vec{B}) \right]$$

define  $u = \frac{1}{8\pi} (E^2 + B^2)$  electromagnetic energy density

$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$  Poynting vector - energy current

then

$$\frac{dE_{\text{mech}}}{dt} = \int_V d^3r \vec{j} \cdot \vec{E} = - \int_V d^3r \left[ \frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} \right]$$

If we define  $E_{EM}$ , the electromagnetic energy of the volume  $V$ , as

$$E_{EM} = \int_V d^3r u$$

then

$$\frac{d}{dt} (E_{\text{mech}} + E_{EM}) = - \oint_S da \hat{n} \cdot \vec{S}$$

or we write  $\frac{dE_{\text{mech}}}{dt} = \int_V d^3r \vec{j} \cdot \vec{E}$  as the rate of change of mechanical energy

or we can write in differential form

$$\vec{j} \cdot \vec{E} + \frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0$$

rate of change of mechanical energy per unit volume  $u_{\text{mech}} = \vec{j} \cdot \vec{E}$

used  $\oint_S da \hat{n} \cdot \vec{S} = \int_V d^3r \vec{\nabla} \cdot \vec{S}$  Gauss' law

local energy conservation law if interpret  $\vec{S}$  as energy current and  $u$  as EM energy density

$$\frac{d}{dt} (\mathcal{E}_{\text{mech}} + \mathcal{E}_{\text{EM}}) = - \oint_S da \hat{n} \cdot \vec{S}$$

total energy in  $V$  can decrease only if electromagnetic energy is being transported through the surface  $S$  by the EM energy current  $\vec{S}$ .

assumes the charged particles do not leave the volume  $V$ .

under certain conditions, we can derive a similar conservation law for the macroscopic maxwell equs.

Consider that  $\vec{j}$  is current of the free <sup>charged</sup> particles.

Then repeating the above steps:

$$\int_V d^3r \vec{j} \cdot \vec{E} = \frac{c}{4\pi} \int d^3r \vec{E} \cdot \left[ \vec{\nabla} \times \vec{H} - \frac{1}{c} \frac{\partial \vec{D}}{\partial t} \right]$$

$$\begin{aligned} \vec{\nabla} \cdot (\vec{E} \times \vec{H}) &= \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H}) \\ &= -\frac{1}{c} \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot (\vec{\nabla} \times \vec{H}) \end{aligned}$$

so

$$\int_V d^3r \vec{j} \cdot \vec{E} = \frac{1}{4\pi} \int_V d^3r \left[ c \vec{\nabla} \cdot (\vec{E} \times \vec{H}) + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right]$$

If the medium is linear, and we have quasi-static conditions, so that

$$\vec{D}(t) \approx \epsilon \vec{E}(t)$$

$$\vec{H}(t) \approx \frac{1}{\mu} \vec{B}(t)$$

then we can write

$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \epsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \epsilon \frac{\partial E^2}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{D})$$

$$\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{\mu} \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2\mu} \frac{\partial B^2}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\vec{B} \cdot \vec{H})$$

\*\* Note: in general, as we will soon see, above conditions are not satisfied,  $\epsilon$  will depend on frequency  $\omega$  and  $\vec{D}(t)$  and  $\vec{E}(t)$  are non locally related in time.  
 $\vec{D}(t) = \int_{-\infty}^t dt' \epsilon(t-t') \vec{E}(t')$ . Only at low frequencies, i.e. in quasistatic case, can we write  $\vec{D}(t) \approx \epsilon(\omega=0) \vec{E}(t)$ .

Assuming the above conditions are met, then

$$\int_V d^3r \vec{j} \cdot \vec{E} + \int_V d^3r \frac{\partial u}{\partial t} = - \oint_S da \hat{n} \cdot \vec{S}$$

$$\text{where } \begin{cases} u = \frac{1}{8\pi} [\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}] \\ \vec{S} = \frac{c}{4\pi} [\vec{E} \times \vec{H}] \end{cases}$$

⇒ electromagnetic energy in dielectric + magnetic materials under quasistatic conditions is

$$\int_V d^3r \left[ \frac{1}{8\pi} \vec{E} \cdot \vec{D} + \frac{1}{8\pi} \vec{B} \cdot \vec{H} \right]$$

↑  
electrostatic  
energy

↑  
magnetostatic  
energy

# Statics

## Electrostatic Energy

Returning to microscopic fields and charges

$$\begin{aligned}\mathcal{E} &= \frac{1}{8\pi} \int_V d^3r E^2 && \text{use } \vec{E} = -\vec{\nabla}\phi \\ &= -\frac{1}{8\pi} \int_V d^3r (\vec{\nabla}\phi) \cdot \vec{E} && \text{use } \vec{\nabla} \cdot (\phi \vec{E}) = \phi \vec{\nabla} \cdot \vec{E} + (\vec{\nabla}\phi) \cdot \vec{E} \\ &= -\frac{1}{8\pi} \int_V d^3r \left[ \vec{\nabla} \cdot (\phi \vec{E}) - \phi \vec{\nabla} \cdot \vec{E} \right] && \text{use } \vec{\nabla} \cdot \vec{E} = 4\pi\rho \\ &= \frac{1}{2} \int_V d^3r \rho \phi - \frac{1}{8\pi} \oint_S da \hat{n} \cdot \phi \vec{E} && \text{by Gauss Theorem}\end{aligned}$$

If let  $V$  be all space,  $S \rightarrow \infty$ , then  $\phi \sim \frac{1}{r}$ ,  $E \sim \frac{1}{r^2}$   
surface integral  $\sim \frac{R^2}{R^3} \rightarrow 0$  as  $R \rightarrow \infty$ .

$$\boxed{\mathcal{E} = \frac{1}{2} \int d^3r \rho \phi}$$

can also use  $\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$  to write

$$\boxed{\mathcal{E} = \frac{1}{2} \int d^3r d^3r' \frac{\rho(\vec{r}) \rho(\vec{r}')}{|\vec{r} - \vec{r}'|}}$$

charge-charge  
interaction

## Magnetostatic Energy

microscopic fields and currents

$$\begin{aligned}\mathcal{E} &= \frac{1}{8\pi} \int d^3r B^2 && \text{use } \vec{B} = \vec{\nabla} \times \vec{A} \\ &= \frac{1}{8\pi} \int d^3r \vec{B} \cdot \vec{\nabla} \times \vec{A} && \text{use } \vec{\nabla} \cdot (\vec{B} \times \vec{A}) = \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \\ &&& \quad - \vec{B} \cdot (\vec{\nabla} \times \vec{A}) \\ &= \frac{1}{8\pi} \int d^3r \left[ \vec{A} \cdot \vec{\nabla} \times \vec{B} - \vec{\nabla} \cdot (\vec{B} \times \vec{A}) \right] && \text{use } \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} \\ &= \frac{1}{2c} \int d^3r \vec{j} \cdot \vec{A} - \frac{1}{8\pi} \int_S da \hat{n} \cdot (\vec{B} \times \vec{A})\end{aligned}$$

as take  $V$  to fill all space,  $S \rightarrow \infty$ , surface term vanishes

$$\boxed{\mathcal{E} = \frac{1}{2c} \int d^3r \vec{j} \cdot \vec{A}}$$

In Coulomb gauge  $\vec{\nabla} \cdot \vec{A} = 0$ ,  $\vec{A}(\vec{r}) = \int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}$

In any other gauge we have  $\vec{A}' = \vec{A} + \vec{\nabla} \chi$   
for some scalar  $\chi$ . So we can always write

$$\vec{A}(\vec{r}) = \int d^3r' \frac{\vec{j}(\vec{r}')}{c |\vec{r} - \vec{r}'|} + \vec{\nabla} \chi$$

regardless of the choice of gauge, where  $\chi$  is then determined so  $\vec{A}$  satisfies the desired gauge condition

$$\mathcal{E} = \frac{1}{2c} \int d^3r d^3r' \frac{\vec{j}(\vec{r}) \cdot \vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|} + \frac{1}{2c^2} \int d^3r \vec{j} \cdot \vec{\nabla} \chi$$

2nd term  $\int d^3r \vec{j} \cdot \vec{\nabla} \chi = \int d^3r \left[ \nabla \cdot (\vec{j} \chi) - \chi \vec{\nabla} \cdot \vec{j} \right]$

$$= \oint_{\mathcal{S}} da \hat{n} \cdot \vec{j} \chi - \int d^3r \chi \vec{\nabla} \cdot \vec{j}$$

vanishes as  $\mathcal{S} \rightarrow \infty$

vanishes in magnetostatics where  $\vec{\nabla} \cdot \vec{j} = 0$

So

$$\mathcal{E} = \frac{1}{2c^2} \int d^3r d^3r' \frac{\vec{j}(\vec{r}) \cdot \vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

current-current interaction

## Momentum Conservation

For charges  $q_i$  at positions  $\vec{r}_i$  with velocities  $\vec{v}_i$

$$\frac{d\vec{P}^{\text{mech}}}{dt} = \sum_i \vec{F}_i = \sum_i q_i (\vec{E}(\vec{r}_i) + \frac{1}{c} \vec{v}_i \times \vec{B}(\vec{r}_i))$$

"mechanical" momentum of the charges  $\uparrow$   
 force on charge  $i$   $\uparrow$   
 $= \int d^3r \left[ \rho \vec{E} + \frac{1}{c} \vec{j} \times \vec{B} \right]$

$$\rho \vec{E} + \frac{1}{c} \vec{j} \times \vec{B} = \frac{1}{4\pi} \left[ \vec{E} (\vec{\nabla} \cdot \vec{E}) + (\vec{\nabla} \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t}) \times \vec{B} \right]$$

Now  $\frac{1}{c} \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) = \frac{1}{c} \left( \frac{\partial \vec{E}}{\partial t} \times \vec{B} \right) + \frac{1}{c} (\vec{E} \times \frac{\partial \vec{B}}{\partial t})$  use  $\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$

$$= \frac{1}{c} \left( \frac{\partial \vec{E}}{\partial t} \times \vec{B} \right) = -\vec{E} \times (\vec{\nabla} \times \vec{E})$$

So  $-\frac{1}{c} \frac{\partial \vec{E}}{\partial t} \times \vec{B} = -\vec{E} \times (\vec{\nabla} \times \vec{E}) - \frac{1}{c} \frac{\partial}{\partial t} (\vec{E} \times \vec{B})$

Therefore

$$\rho \vec{E} + \frac{1}{c} \vec{j} \times \vec{B} = \frac{1}{4\pi} \left[ \vec{E} (\vec{\nabla} \cdot \vec{E}) + \vec{B} (\vec{\nabla} \cdot \vec{B}) - \vec{B} \times (\vec{\nabla} \times \vec{B}) - \vec{E} \times (\vec{\nabla} \times \vec{E}) - \frac{1}{c} \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) \right]$$

Define electromagnetic momentum density

$$\vec{\Pi} = \frac{1}{4\pi c} \vec{E} \times \vec{B} = \frac{1}{c^2} \vec{S} \quad (\vec{S} \text{ is Poynting vector})$$

then

$$\frac{d\vec{P}^{\text{mech}}}{dt} + \frac{d}{dt} \int_V d^3r \vec{\Pi} = \frac{1}{4\pi} \int_V d^3r \left[ \vec{E} (\vec{\nabla} \cdot \vec{E}) - \vec{E} \times (\vec{\nabla} \times \vec{E}) + \vec{B} (\vec{\nabla} \cdot \vec{B}) - \vec{B} \times (\vec{\nabla} \times \vec{B}) \right]$$

want to rewrite as a surface integral



$i$ th component of integrand on right hand side is ( $\vec{E}$  part only)  
(sum over repeated indices)

$$E_i \partial_j E_j - \epsilon_{ijk} E_j \epsilon_{klm} \partial_l E_m$$

$$= E_i \partial_j E_j - (\delta_{cl} \delta_{jm} - \delta_{cm} \delta_{jl}) E_j \partial_l E_m$$

$$= E_i \partial_j E_j - E_j \partial_i E_j + E_j \partial_j E_i$$

$$= \partial_j (E_i E_j - \frac{1}{2} \delta_{ij} E^2)$$

Define Maxwell's stress tensor

$$T_{ij} = \frac{1}{4\pi} [E_i E_j + B_i B_j - \frac{1}{2} \delta_{ij} (E^2 + B^2)]$$

(note  $T_{ij} = T_{ji}$   
Symmetric tensor)

Then

$$\frac{d}{dt} \vec{p}^{\text{mech}}_i + \frac{d}{dt} \int_V d^3r \Pi_i = \int_V d^3r \partial_j T_{ij}$$

$$\left( \partial_j T_{ij} = \frac{\partial T_{ij}}{\partial x_j} \right)$$

$$= \oint_S da T_{ij} \hat{m}_j$$

$$\frac{d}{dt} \vec{p}^{\text{mech}} + \frac{d}{dt} \int_V d^3r \vec{\Pi} = \oint_S da \vec{T} \cdot \hat{m}$$

-  $T_{ij}$  gives the flow of the  $i$ th component of electromagnetic field momentum through an element of surface area  $\perp$  to direction  $\hat{e}_j$

For static situations where  $\frac{d\Pi}{dt} = 0$ ,  $\frac{d\vec{p}^{\text{mech}}}{dt} = \vec{F}^{\text{tot}} = \oint_S da \vec{T} \cdot \hat{m}$   
gives electromagnetic force on the surface  $S$

Note:  $\frac{d\vec{P}^{\text{mech}}}{dt}$  is ~~also~~ equal to the total electromagnetic force on the volume  $V$ .

Hence we can write

$$\vec{F}_{\text{EM}} = \oint_S da \vec{T} \cdot \hat{n} - \frac{d}{dt} \int_V d^3r \vec{\Pi}$$

for static situations, the 2<sup>nd</sup> term vanishes and

$$\vec{F}_{\text{EM}} = \oint_S da \vec{T} \cdot \hat{n} \quad \vec{T}_{ij} \text{ is } i^{\text{th}} \text{ component of static force on unit area with normal } \hat{e}_j$$

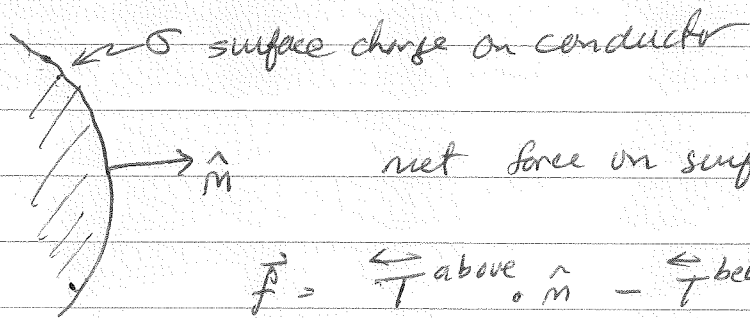
this is origin of the term "stress" tensors.

$\vec{T}$  is like the stress tensor of an elastic medium.

$T_{xx}, T_{yy}, T_{zz}$  are like pressure.

off diagonal elements are like shear stresses

## Force on a conductor surface,



net force on surface per unit area is

$$\vec{f} = \vec{T}_{\text{above}} \cdot \hat{n} - \vec{T}_{\text{below}} \cdot \hat{n}$$

$\uparrow = 0$  as  $\vec{E} = 0$  inside conductor

$$\vec{f} = \frac{1}{4\pi} \left[ \vec{E} (\vec{E} \cdot \hat{n}) - \frac{1}{2} \hat{n} E^2 \right]$$

for conducting surface

$$\hat{n} \cdot \vec{E}^{\text{above}} = 4\pi\sigma \quad (\text{since } \vec{E}^{\text{below}} = 0)$$

and tangential component  $\vec{E} = 0$

$$\Rightarrow \vec{E} = 4\pi\sigma \hat{n}$$

$$\text{So } \vec{f} = \frac{1}{4\pi} \left[ (4\pi\sigma \hat{n}) (4\pi\sigma) - \frac{1}{2} \hat{n} (4\pi\sigma)^2 \right]$$

$$\boxed{\vec{f} = \frac{1}{4\pi} \left[ (4\pi\sigma)^2 \hat{n} - \frac{1}{2} \hat{n} (4\pi\sigma)^2 \right]}$$

$$\vec{f} = \frac{\hat{n}}{4\pi} \left[ (4\pi\sigma)^2 - \frac{1}{2} (4\pi\sigma)^2 \right] = 2\pi\sigma^2 \hat{n}$$

force per unit area

$$\boxed{\vec{f} = 2\pi\sigma^2 \hat{n} = \frac{1}{2} \sigma \vec{E}}$$

$$\vec{f} = \sigma \vec{E}_{\text{ave}}$$

where  $\vec{E}_{\text{ave}} = \frac{1}{2} (\vec{E}^{\text{above}} + \vec{E}^{\text{below}})$   
is average field at surface  
averaging over above + below

Note factor  $\frac{1}{2}$ . Naively one might have thought  $\vec{f} = \sigma \vec{E}$ . But need to exclude self field of charge on surface from acting on itself, see also Jackson pg 42 for another approach.