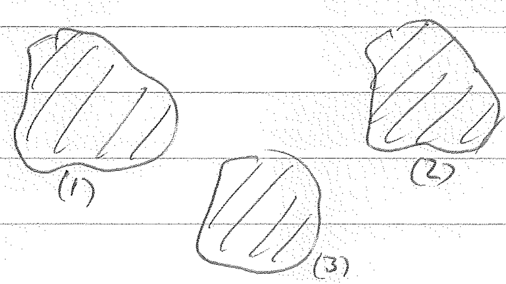
Capacitance

Consider a set of conductors with potential $\phi(\vec{r}) = V_i$ fixed on conductor i



(also need condition on $\phi(\vec{r}) \to \infty$ if system is not enclosed)

From uniqueness theorem we know that specifying the V_i on each conductor is enough to determine the potential $\phi(\vec{r})$ everywhere. We can write this potential in the following form.

Let $\phi^{(i)}(\vec{r})$ be the solution to the boundary value problem

$$\nabla^2 \phi^{(i)}(r) = 0 \quad \text{and} \quad \phi^{(i)}(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r} \text{ on surface of conductor}(i) \\ 0 & \text{if } \vec{r} \text{ on surface of any other conductor}(j),\ j \neq i \end{cases}$$

Then by superposition

$$\phi(\vec{r}) = \sum_i V_i \, \phi^{(i)}(\vec{r})$$

is solution to the problem $\nabla^2 \phi = 0$ and $\phi(\vec{r}) = V_i$ for $\vec{r}$ on surface of conductor (i)

The surface charge density at $\vec{r}$ on surface of conductor (i) is

$$\sigma^{(i)}(\vec{r}) = \frac{-1}{4\pi} \frac{\partial \phi(\vec{r})}{\partial n} = -\frac{1}{4\pi} \sum_j V_j \frac{\partial \phi^{(j)}(\vec{r})}{\partial n}$$

where $\dfrac{\partial \phi}{\partial n} = (\vec{\nabla} \phi) \cdot \hat{n}$ is the derivative normal to the surface at point $\vec{r}$.

The total charge on conductor (i) is

$$Q_i = \int_{S_i} da\, \sigma^{(i)}(\vec{r}) = -\frac{1}{4\pi}\sum_j V_j \int_{S_i} da \frac{\partial \phi^{(j)}}{\partial n}$$

surface of conductor(i)

Define $C_{ij} \equiv -\frac{1}{4\pi}\int_{S_i} da \frac{\partial \phi^{(j)}}{\partial n}$ the C_{ij} depend only on the geometry of the conductors

Then we have

$$\boxed{Q_i = \sum_j C_{ij} V_j}$$

C_{ij} is the capacitance matrix

The charge on conductor (i) is a linear function of the potentials V_j on the conductors (j)

Since we know that specifying the Q_i that is on each conductor will uniquely determine $\phi(\vec{r})$ and hence the potential V_i on each conductor, the capacitance matrix is invertable

$$V_i = \sum_j [C^{-1}]_{ij} Q_j$$

The electrostatic energy of the conductors is then

$$\mathcal{E} = \frac{1}{2}\int d^3r\, \rho\, \phi = \frac{1}{2}\sum_i Q_i V_i = \frac{1}{2}\sum_{i,j} C_{ij} V_i V_j = \frac{1}{2}\sum_{ij} C^{-1}_{ij} Q_i Q_j$$

$$[V\cdot C\cdot V] \qquad [Q\cdot C^{-1}\cdot Q]$$

Common to define capacitance of two conductors by

$$C = \frac{Q}{V_1 - V_2}$$

when conductor (1) has charge Q
conductor (2) has charge $-Q$
$V_1 - V_2$ is potential difference between the two conductors.
all other conductors fixed at $V_i = 0$

We can determine C in terms of the elements of the matrix C_{ij}

$$\left.\begin{aligned} Q &= C_{11} V_1 + C_{12} V_2 \\ -Q &= C_{21} V_1 + C_{22} V_2 \end{aligned}\right\} \Rightarrow V_2 = -\left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}}\right) V_1$$

$$\Rightarrow Q = \left[C_{11} - C_{12} \left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right) \right] V_1$$

$$V_1 - V_2 = \left[1 + \left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right) \right] V_1$$

$$C = \frac{Q}{V_1 - V_2} = \frac{C_{11} - C_{12}\left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}}\right)}{1 + \left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}}\right)}$$

$$\boxed{C = \frac{C_{11} C_{22} - C_{12} C_{21}}{C_{11} + C_{12} + C_{21} + C_{22}}}$$

Capacitance can also be defined when the space between the conductors is filled with a dielectric ϵ. In this case, if Q_i is the free charge, then Q_i/ϵ is the effective total charge to use in computing ϕ.

$$\Rightarrow \frac{Q_i}{\epsilon} = \sum_j C_{ij}^{(0)} V_j$$ where $C_{ij}^{(0)}$ are capacitances appropriate to a vacuum between the conductors

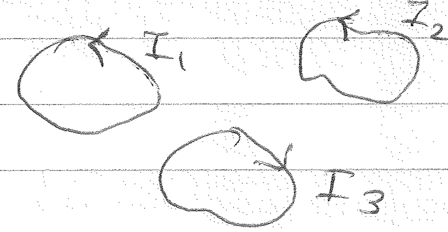
$$\Rightarrow Q_i = \sum_j \epsilon\, C_{ij}^{(0)} V_j$$

$$= \sum_j C_{ij} V_j \quad \text{where } C_{ij} = \epsilon\, C_{ij}^{(0)}$$

the capacitance is increased by a factor the dielectric constant ϵ.

Inductance

Consider a set of current carrying loops C_i with currents I_i



In Coulomb gauge, we can write the magnetic vector potential $\vec{A}$ from these current loops as

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \sum_i \frac{I_i}{c} \oint_{C_i} \frac{d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

↑ integrate over loop C_i
integration variable is $\vec{r}'$

The magnetic flux through loop i is

$$\Phi_i = \int_{S_i} da\, \hat{n}\cdot\vec{B} = \int_{S_i} da\, \hat{n}\cdot\vec{\nabla}\times\vec{A} = \oint_{C_i} d\vec{l}\cdot\vec{A}$$

↑ surface bounded by loop C_i

$$\Phi_i = \sum_j \frac{I_j}{c} \oint_{C_i}\oint_{C_j} \frac{d\vec{l}\cdot d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

pure geometrical quantity ↓

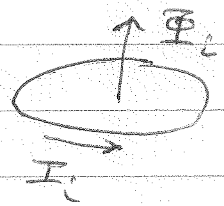
$$\boxed{\Phi_i \equiv c\sum_j M_{ij} I_j}$$

where $M_{ij} = \oint_i\oint_j \frac{d\vec{l}_i\cdot d\vec{l}_j}{c^2|\vec{r}-\vec{r}'|}$

is the <u>mutual</u> inductance of loops (i) and (j). $M_{ij} = M_{ji}$

$L_i \equiv M_{ii}$ is <u>self-inductance</u> of loop (i)

The sign convention in the above is that Φ_i is computed in direction given by right hand rule, according to the direction taken for current in loop (i)



Magnetostatic energy

$$\mathcal{E} = \frac{1}{2c}\int d^3r\ \vec{j}\cdot\vec{A} = \frac{1}{2c}\sum_i \oint_{C_i} d\vec{l}\cdot\vec{A}\ I_i$$

$$= \frac{1}{2c}\sum_i \Phi_i I_i$$

$$\mathcal{E} = \frac{1}{2}\sum_{i,j} M_{ij} I_i I_j$$

Electromagnetic Waves in a vacuum

No sources $\vec{J}=0$, $\rho=0$

1) $\vec{\nabla}\cdot\vec{E}=0$ 3) $\vec{\nabla}\cdot\vec{B}=0$

2) $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{c\,\partial t}$ 4) $\vec{\nabla}\times\vec{B}=\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

$$\nabla\times(2) \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \underbrace{\vec{\nabla}(\vec{\nabla}\cdot\vec{E})}_{0 \text{ by } (1)} - \nabla^2\vec{E} = -\frac{\partial}{c\,\partial t}(\vec{\nabla}\times\vec{B})$$

$$-\nabla^2 E = \frac{-\partial}{c\,\partial t}(\vec{\nabla}\times\vec{B}) = \frac{-\partial}{c\,\partial t}\left(\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)$$

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

Similarly

$$\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0$$

wave equation
wave speed is c.

Note: in MKS units, above wave equation looks like

$$\nabla^2\vec{E} - \varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

It was noticed that the speed of electromagnetic wave,

$\frac{1}{\sqrt{\varepsilon_0\mu_0}} = 3\times10^8$ m/s was the same as the speed of light! This observation was a key element in showing that light was in fact electromagnetic wave

Harmonic

Plane waves

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$
$$\vec{B}(\vec{r},t) = \mathrm{Re}\left[\vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

complex exponential form

$\vec{k}$ is wave vector

ω is angular frequency

$\nu = \omega/2\pi$ is frequency

$T = 1/\nu$ is period

$\lambda = \frac{2\pi}{|k|}$ is wavelength

$|\vec{E}_k|$, $|B_k|$ is amplitude

$\vec{E}(\vec{r}+\lambda\hat{k},t) = \vec{E}(\vec{r},t)$ periodic in space in direction $\hat{k}$ with period λ

$\vec{E}(\vec{r},t+T) = \vec{E}(r,t)$ periodic in time with period T

"plane wave" $\Rightarrow$ $\vec{E}(\vec{r},t)$ is constant in space on planes with normal $\hat{n} \parallel \vec{k}$.

properties of EM plane waves

$$\vec{\nabla}\cdot\vec{E}=0 \Rightarrow \mathrm{Re}\left[\vec{E}_k\cdot\vec{\nabla}e^{i(k\cdot\vec{r}-\omega t)}\right]$$
$$= \mathrm{Re}\left[i\vec{E}_k\cdot\vec{k}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$
$$\Rightarrow \vec{E}_k\cdot\vec{k}=0$$

amplitude is orthogonal to $\vec{k}$

$$\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \vec{B}_k\cdot\vec{k}=0$$

amplitude orthogonal to $\vec{k}$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \quad Re\left[\vec{\nabla} \times \vec{B}_k e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right] = Re\left[\frac{1}{c} \vec{E}_k \frac{\partial}{\partial t} e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right]$$

$$\Rightarrow Re\left[-\vec{B}_k \times \vec{\nabla} e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right] = Re\left[-\frac{i\omega}{c} \vec{E}_k e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right]$$

$$\Rightarrow \quad Re\left[i\vec{k} \times \vec{B}_k e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right] = Re\left[-\frac{i\omega}{c} \vec{E}_k e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right]$$

$$\Rightarrow \quad \vec{k} \times \vec{B}_k = -\frac{\omega}{c} \vec{E}_k$$

$$\vec{k} \times \vec{k} \times \vec{B}_k = -k^2 \vec{B}_k = -\frac{\omega}{c} \vec{k} \times \vec{E}_k$$

$$\underline{\vec{B}_k = \frac{\omega}{ck^2} \vec{k} \times \vec{E}_k} = \frac{\omega}{ck} \hat{k} \times \vec{E}_k$$

Finally

$$\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\Rightarrow \quad Re\left[\vec{E}_k \nabla^2 e^{i(\vec{k}\cdot\vec{r} - \omega t)} - \frac{\vec{E}_k}{c^2} \frac{\partial^2}{\partial t^2} e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right] = 0$$

$$\Rightarrow \quad Re\left[\vec{E}_k (-k^2) e^{i(\vec{k}\cdot\vec{r} - \omega t)} + \frac{\omega^2}{c^2} \vec{E}_k e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right] = 0$$

$$\Rightarrow \quad k^2 = \frac{\omega^2}{c^2}$$

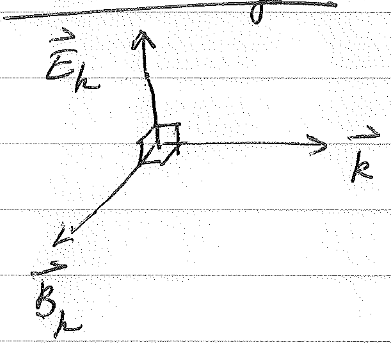
$$\boxed{\omega = \pm kc}$$ dispersion relation

insert in above

$$\vec{B}_k = \hat{k} \times \vec{E}_k \qquad \hat{k} = \frac{\vec{k}}{|\vec{k}|}$$

$$\Rightarrow \quad |\vec{B}_k| = |\vec{E}_k|$$

Summary



$\vec{E}_k \perp \vec{k}$
$\vec{B}_k \perp \vec{k}$ } "transverse" polarization

$\vec{B}_k = \hat{k} \times \vec{E}_k$

$\omega^2 = c^2 k^2$

$|\vec{B}_k| = |\vec{E}_k| \Rightarrow$ Lorentz force from plane EM wave on charge q is

$$q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

magnetic force is smaller factor $\left(\frac{v}{c}\right)$ as compared to electric force - can usually be ignored

Most general solution is a linear superposition of the above harmonic plane waves

$$\vec{E}(\vec{r}, t) = \int \frac{d^3k}{(2\pi)^3} \vec{E}_k \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$ Fourier transform

$\vec{E}(\vec{r}, t)$ is real $\Rightarrow$ $\vec{E}_k^{\,*} = \vec{E}_{-k}$

For dispersion relation $\omega^2 = c^2 k^2$ we can write

$$\vec{k}\cdot\vec{r} - \omega t = \vec{k}\cdot(\vec{r} - \vec{v}t)$$

where $\vec{v} = c\hat{k}$ is velocity of wave
If we only combine waves traveling in same direction $\hat{k}$, then

$$\vec{E}(\vec{r}, t) = \int \frac{d^3k}{(2\pi)^3} \vec{E}_k \, e^{i\vec{k}\cdot(\vec{r} - \vec{v}t)} = \vec{E}(\vec{r} - \vec{v}t, 0)$$

The general plane wave solution of wave equation always has this property

$$\vec{E}(r, t) = \vec{E}(\vec{r} - \vec{v}t, 0)$$ If know $\vec{E}$ at $t=0$, then know $\vec{E}$ at all times t