

Energy & momentum in EM wave

$$\left. \begin{aligned} \vec{E} &= \text{Re} [\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}] = \vec{E}_k \cos(\vec{k}\cdot\vec{r}-\omega t) \\ \vec{B} &= \text{Re} [\vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}] = \hat{k} \times \vec{E}_k \cos(\vec{k}\cdot\vec{r}-\omega t) \end{aligned} \right\} \begin{array}{l} \text{for} \\ \text{real} \\ \vec{E}_k \end{array}$$

energy density $u = \frac{1}{8\pi} (E^2 + B^2)$

$$= \frac{1}{8\pi} [E_k^2 + E_k^2] \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$= \frac{1}{4\pi} E_k^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

Poynting vector

energy current

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$$

$$= \frac{c}{4\pi} [\vec{E}_k \times (\hat{k} \times \vec{E}_k)] \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$= \frac{c}{4\pi} \hat{k} E_k^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$\vec{S} = c u \hat{k}$$

momentum density $\vec{\pi} = \frac{1}{c^2} \vec{S} = \frac{u}{c} \hat{k}$

$$u = c |\vec{\pi}| \quad - \text{energy momentum relation of photons!}$$

For visible light $\lambda \sim 5 \times 10^{-7} \text{ m} \sim 5000 \text{ \AA}$

$$T = \frac{\lambda}{c} = 1.6 \times 10^{-15} \text{ sec}$$

most classical measurements on macroscopic scales $t \gg T$, $l \gg \lambda$

measure average quantities

$$\langle u \rangle = \frac{1}{T} \int_0^T dt u = \frac{1}{8\pi} E_k^2 \quad \text{as } \langle \cos^2 \theta \rangle = \frac{1}{2}$$

$$\langle \vec{S} \rangle = c \langle u \rangle \hat{k}$$

$$\langle \vec{\Pi} \rangle = \frac{1}{c} \langle u \rangle \hat{k}$$

intensity = average power per area transported by wave through surface with normal \hat{n}

$$I = \langle \vec{S} \rangle \cdot \hat{n}$$

Electromagnetic waves in matter

Macroscopic Maxwell equations with no sources, $\rho=0$, $\vec{j}=0$.
("free" charge and current vanishes)

$$\begin{aligned}\vec{\nabla} \cdot \vec{D} &= 0 & \vec{\nabla} \times \vec{H} &= \frac{1}{c} \frac{\partial \vec{D}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 & \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{c \partial t}\end{aligned}$$

linear materials

$$\begin{aligned}\vec{B} &= \mu \vec{H} \\ \vec{D} &= \epsilon \vec{E}\end{aligned}$$

if μ and ϵ were simply constants then the above would become

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= 0 & \vec{\nabla} \times \vec{B} &= \frac{\mu \epsilon}{c} \frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 & \vec{\nabla} \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}\end{aligned}$$

Then

$$\begin{aligned}\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) &= -\nabla^2 \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) \\ &= -\frac{1}{c} \frac{\partial}{\partial t} \left(\frac{\mu \epsilon}{c} \frac{\partial \vec{E}}{\partial t} \right) \\ &= -\frac{\mu \epsilon}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}\end{aligned}$$

wave equation with wave speed $\frac{c}{\sqrt{\mu \epsilon}} < c$

This would be very much as for waves in a vacuum, except for the following minus

changes:

$$\omega^2 = \frac{c^2 k^2}{\mu \epsilon}$$

dispersion relation
changed by constant
factor

$$\begin{aligned}\vec{E}_k &\perp \vec{k} \\ \vec{B}_k &\perp \vec{k}\end{aligned}$$

$$i \vec{k} \times \vec{E}_k = i \frac{\omega}{c} \vec{B}_k$$

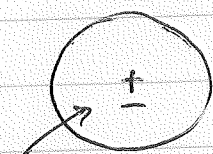
$$\frac{c |\vec{k}|}{\omega} \hat{k} \times \vec{E}_k = \vec{B}_k$$

$$\Rightarrow \sqrt{\mu \epsilon} \hat{k} \times \vec{E}_k = \vec{B}_k \quad |\vec{B}_k| > |\vec{E}_k|$$

wave speed $v = \frac{c}{\sqrt{\mu \epsilon}} < c$

In general however things are much more complicated
because ϵ cannot be viewed as a constant
when considering time varying behavior!

Time dependent polarizability of an atom



electron cloud

If displace center of electron cloud by a distance \vec{r} , there is a restoring

$$\text{force } \vec{F}_{\text{rest}} = -\frac{e^2 \vec{r}}{4\pi R^3} = -m\omega_0^2 \vec{r}$$

↑ ↑
electron mass resonant frequency

Also, in general there will be a damping force

$$\vec{F}_{\text{damp}} = -m\gamma \frac{d\vec{r}}{dt}$$

due to transfer of energy from atom to other degrees of freedom.

In an external electric field $\vec{E}(t)$, the equation of motion for electron cloud is

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F}_{\text{tot}} = -e \vec{E}(t) - m\omega_0^2 \vec{r} - m\gamma \frac{d\vec{r}}{dt}$$

$$\frac{d^2 \vec{r}}{dt^2} + \gamma \frac{d\vec{r}}{dt} + \omega_0^2 \vec{r} = -\frac{e \vec{E}(t)}{m}$$

assuming \vec{E} is spatially constant over atomic distances

For harmonic oscillation $\vec{E}(t) = \vec{E}_0 e^{-i\omega t}$

Assume solution $\vec{r}(t) = \vec{r}_0 e^{-i\omega t}$

(in the end, we will take the real parts)

Substitute into equation of motion

$$-\omega^2 \vec{r}_0 - i\omega\gamma \vec{r}_0 + \omega_0^2 \vec{r}_0 = -\frac{e \vec{E}_0}{m}$$

$$\vec{r}_0 = \frac{-e}{m(\omega_0^2 - \omega^2 - i\omega\gamma)} \vec{E}_0$$

polarization

$$\vec{p} = -e\vec{r} = \vec{p}_0 e^{-i\omega t}$$

$$\vec{p}_0 = \frac{e^2}{m} \frac{1}{(\omega_0^2 - \omega^2 - i\omega\gamma)} \vec{E}_0 = \alpha(\omega) \vec{E}_0$$

$$\alpha(\omega) = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} \quad \text{freq dependent polarizability}$$

Since α is complex the polarization does not in general oscillate in phase with \vec{E} .

If $\alpha(\omega) = |\alpha| e^{i\delta}$ δ is phase of complex α

$$\alpha = \alpha_1 + i\alpha_2 \quad \text{then } |\alpha| = \sqrt{\alpha_1^2 + \alpha_2^2} \quad \tan \delta = \alpha_2 / \alpha_1$$

$$\vec{p}(t) = \alpha(\omega) \vec{E}(t) = |\alpha| e^{i\delta} \vec{E}_0 e^{-i\omega t} = |\alpha| \vec{E}_0 e^{-i(\omega t - \delta)}$$

↑
phase shifted by δ

For a general electric field

$$\vec{E}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \vec{E}_\omega e^{-i\omega t}$$

$$\vec{E}_\omega^* = \vec{E}_{-\omega}$$

$$\vec{p}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \vec{p}_\omega e^{-i\omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega) \vec{E}_\omega e^{-i\omega t}$$

Substitute in $\vec{E}_\omega = \int_{-\infty}^{\infty} dt' \vec{E}(t') e^{i\omega t'}$ to get

$$\vec{p}(t) = \int_{-\infty}^{\infty} dt' \vec{E}(t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega) e^{-i\omega(t-t')}$$

$$\vec{p}(t) = \int_{-\infty}^{\infty} dt' \vec{E}(t') \tilde{\alpha}(t-t')$$

↑ Fourier transf of $\alpha(\omega)$

\vec{p} at time t is due to \vec{E} at all times t'
non local in time

By superposition
the response to
component \vec{E}_ω of
 $\vec{E}(t)$ is just
 $\vec{p}_\omega = \alpha(\omega) \vec{E}_\omega$

$\tilde{\alpha}(t)$ is the response to $\vec{E}(t) = \delta(t)$

For our simple model

$$\tilde{\alpha}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

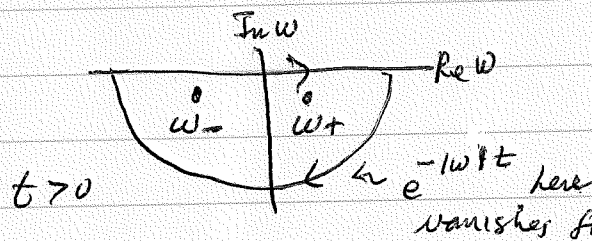
do by contour integration

$$\frac{1}{\omega^2 + i\gamma\omega - \omega_0^2} = \frac{1}{(\omega - \omega_+) (\omega - \omega_-)}$$

$$\omega_{\pm} = -\frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \gamma^2/4} = -\frac{i\gamma}{2} \pm \bar{\omega}$$

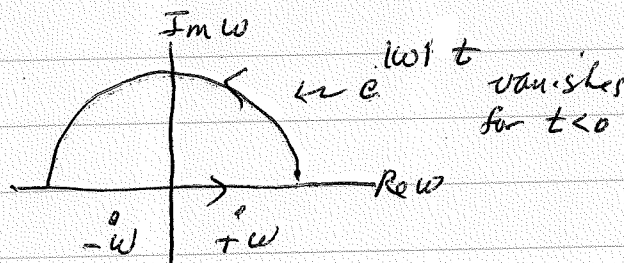
poles at ω_{\pm} are in lower half complex plane.

for $t > 0$, close contour in lower half plane



contour encloses poles
+ get contribution
vanishes for $t > 0$

for $t < 0$, close contour in upper half plane



contour encloses
no poles \Rightarrow integral
vanishes

$$\tilde{\alpha}(t) = 0 \quad \text{for } t < 0$$

Causal response! No polarization until electric field turns on

For $t > 0$

minus sign since we go around contour clockwise

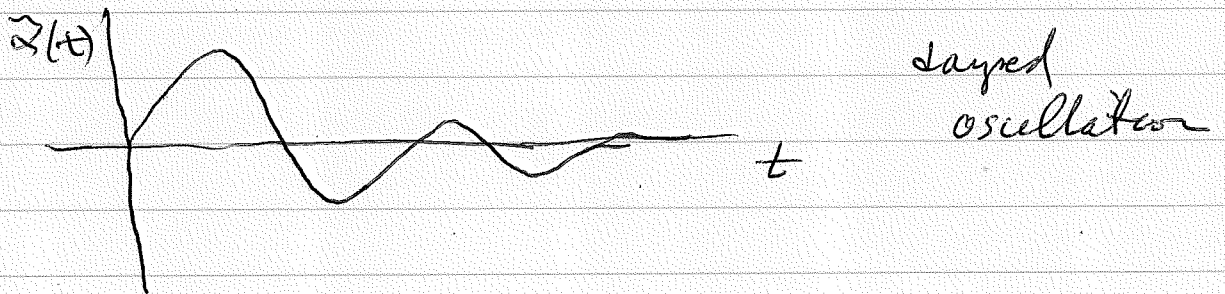
$$\tilde{\alpha}(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{e^2}{m} \frac{(-1)}{(\omega - \omega_+)(\omega - \omega_-)}$$

from residue theorem

$$= (-2\pi i) \frac{e^2}{m} \frac{(-1)}{2\pi} \left[\frac{e^{-i\omega_+ t}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_- t}}{\omega_- - \omega_+} \right]$$

$$= \frac{ie^2}{m} \left[\frac{e^{-\gamma t/2} e^{-i\bar{\omega} t}}{2\bar{\omega}} - \frac{e^{-\gamma t/2} e^{i\bar{\omega} t}}{2\bar{\omega}} \right]$$

$$\tilde{\alpha}(t) = \begin{cases} \frac{e^2}{m} \frac{e^{-\gamma t/2}}{\bar{\omega}} \sin(\bar{\omega} t) & t > 0 \\ 0 & t < 0 \end{cases}$$



Polarization density $\vec{P}_\omega = 4\pi X(\omega) \vec{E}_\omega$ for harmonic oscillation

$X(\omega) \approx n \alpha(\omega)$ for dilute system

\uparrow atom density
can use Clausius - Mossotti correction for denser materials

$$\Rightarrow \vec{D}_\omega = \epsilon(\omega) \vec{E}_\omega \quad \epsilon(\omega) = 1 + 4\pi X(\omega)$$

\uparrow freq dependent

→ as with \vec{f} and \vec{E} , relation between \vec{D} and \vec{E} is non-local in time

$$\vec{D}(t) \neq \epsilon \vec{E}(t)$$

rather

$$\vec{D}(t) = \int_{-\infty}^{\infty} dt' \vec{E}(t') \tilde{\epsilon}(t-t')$$

↳ Fourier transf of $\epsilon(\omega)$

Ampere's law is

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{f} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}$$

becomes $\frac{1}{\mu} \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{f} + \frac{1}{c} \int_{-\infty}^{\infty} dt' \vec{E}(t') \frac{d}{dt} \tilde{\epsilon}(t-t')$

↳ integro-differential equation!

Maxwell's equations only look simple when expressed in terms of Fourier transforms

$$\vec{E}(\vec{r}, t) = \vec{E}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\vec{B}(\vec{r}, t) = \vec{B}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\vec{D}(\vec{r}, t) = \vec{D}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\vec{H}(\vec{r}, t) = \vec{H}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

Maxwell's Equ for source free system $\rho = \vec{f} = 0$

$$\vec{\nabla} \cdot \vec{D} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{c \partial t}, \quad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{c \partial t}$$