

## EM waves in Conductors

Conduction electrons are mobile, not bound  
⇒ we have to include the  $\vec{j}_f$  and  $j_f$  from them.

Simple classical model for electron motion - "Drude" Model

$$m \ddot{\vec{r}} = -e \vec{E}(t) - \frac{m}{\tau} \dot{\vec{r}} \quad \text{no restoring force!}$$

$\uparrow$  external  $\uparrow$  damping force due to collisions  
 $E$  field  $\tau$  is "relaxation time"

$$\vec{E} = \vec{E}_\omega e^{-i\omega t} \Rightarrow \vec{r} = \vec{r}_\omega e^{-i\omega t} \quad \text{solution}$$

plug in to get

$$(-\omega^2 - \frac{i\omega}{\tau}) \vec{r}_\omega = -\frac{e}{m} \vec{E}_\omega \Rightarrow \vec{r}_\omega = \frac{e}{m} \frac{1}{\omega^2 + \frac{i\omega}{\tau}} \vec{E}_\omega$$

current is

$$\vec{j}_f = -me \dot{\vec{r}}, \quad \vec{j}_f = j_\omega e^{-i\omega t} \Rightarrow j_\omega = -en(-i\omega) \vec{r}_\omega$$

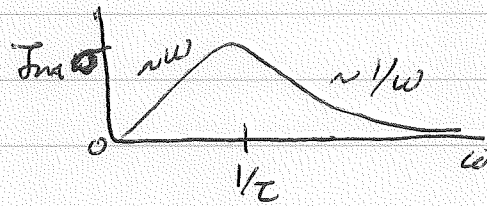
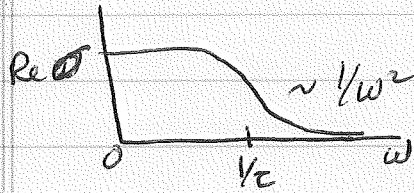
density of electrons

$$j_\omega = \frac{ne^2 \tau \omega}{m (\omega^2 + \frac{i\omega}{\tau})} \vec{E}_\omega = \frac{me^2 \tau}{m} \frac{1}{1 - i\omega \tau} \vec{E}_\omega$$

$$j_\omega = \sigma(\omega) \vec{E}_\omega$$

conductivity

$$\sigma(\omega) = \frac{me^2 \tau}{m} \frac{1}{1 - i\omega \tau}$$



$$\text{Re } \sigma = \frac{\sigma_0}{1 + \omega^2 \tau^2}$$

$$\text{Im } \sigma = \frac{\sigma_0 \omega \tau}{1 + \omega^2 \tau^2}$$

$$\sigma_0 = \sigma(0) = \frac{ne^2 \tau}{m}$$

dc conductivity

Charge density  $\rho_f$  given by charge conservation law,  
for plane waves

$$\rho_f = \rho_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}, \quad \vec{j}_f = \vec{j}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\frac{\partial \rho_f}{\partial t} = -\vec{\nabla} \cdot \vec{j}_f \Rightarrow -i\omega \rho_0 = -i\vec{k} \cdot \vec{j}_0$$

$$\rho_0 = \frac{\vec{k} \cdot \vec{j}_0}{\omega} = \frac{\sigma(\omega) \vec{k} \cdot \vec{E}_0}{\omega}$$

### Maxwell Equations

$$1) \quad \vec{\nabla} \cdot \vec{D} = 4\pi \rho_f$$

$$2) \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$3) \quad \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$4) \quad \vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{j}_f + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}$$

Assume  $\vec{H} = \vec{B}/\mu$ ,  $\mu$  constant

$$\vec{D}_\omega = \epsilon_b(\omega) \vec{E}_\omega$$

$\epsilon_b(\omega)$  is dielectric function

$$\vec{j}_\omega = \sigma(\omega) \vec{E}_\omega$$

from the bound charges

$$j_\omega = \frac{\sigma}{\omega} \vec{k} \cdot \vec{E}_\omega$$

$\sigma(\omega)$  is conductivity from free charges

For harmonic plane wave solutions  $\vec{E} = E_{\omega} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$   
etc.

$$1) \Rightarrow i\vec{k} \cdot \vec{D}_{\omega} = i\vec{k} \cdot \epsilon_b \vec{E}_{\omega} = 4\pi j_{\omega} = 4\pi \sigma \frac{\vec{k} \cdot \vec{E}_{\omega}}{\omega}$$

$$\Rightarrow i\vec{k} \cdot \vec{E}_{\omega} \left( \epsilon_b + \frac{4\pi i \sigma}{\omega} \right) = 0$$

$$2) \Rightarrow i\mu \vec{k} \cdot \vec{H}_{\omega} = 0$$

$$3) \Rightarrow i\vec{k} \times \vec{E}_{\omega} = \frac{i\omega \vec{B}_{\omega}}{c} = \frac{i\omega \mu \vec{H}_{\omega}}{c}$$

$$\begin{aligned} 4) \Rightarrow i\vec{k} \times \vec{H}_{\omega} &= \frac{4\pi}{c} \vec{j}_{\omega} - \frac{i\omega}{c} \vec{D}_{\omega} \\ &= \frac{4\pi \sigma}{c} \vec{E}_{\omega} - \frac{i\omega}{c} \epsilon_b \vec{E}_{\omega} \\ &= -\frac{i\omega}{c} \left( \epsilon_b + \frac{4\pi i \sigma}{\omega} \right) \vec{E}_{\omega} \end{aligned}$$

Notice: all the equations above look exactly like what we had for the dielectric, provided we define

$$\boxed{\epsilon(\omega) = \epsilon_b(\omega) + \frac{4\pi i \sigma(\omega)}{\omega}}$$

So all results for the dielectric case carry over to conductors, provided we make the above substitution. In particular

dispersion relation for transverse modes  $\boxed{k^2 = \frac{\omega^2}{c^2} \epsilon(\omega)}$

The main difference between dielectrics + conductors has to do with the contribution that the  $4\pi i\sigma/\omega$  makes to the real and imaginary parts of  $\epsilon(\omega)$ .

For single Drude model  $\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}$       $\sigma_0 = \frac{ne^2\tau}{m}$

① Low frequencies  $\omega \ll 1/\tau$ ,  $\omega \ll \omega_0$

$\epsilon_b(\omega) \approx \epsilon_b(0)$  real

$\sigma(\omega) \approx \sigma_0$  real

↑ resonant freq of bound electrons

⇒  $\epsilon(\omega) \approx \epsilon_b(0) + \frac{4\pi i\sigma_0}{\omega}$

← gives large  $\epsilon_2$  as  $\omega \rightarrow 0$

⇒ strong dissipation

$\text{Re } \epsilon = \epsilon_1$

$\text{Im } \epsilon = \epsilon_2$

when  $\frac{\epsilon_2}{\epsilon_1} = \frac{4\pi\sigma_0}{\omega\epsilon_b(0)} \gg 1$

we call this regime a "good" conductor.

conduction electrons dominate the response - waves strongly attenuated

when  $\frac{\epsilon_2}{\epsilon_1} = \frac{4\pi\sigma_0}{\omega\epsilon_b(0)} \ll 1$

we call this regime a "poor" conductor.

little absorption of energy by conduction electrons. waves propagate

one always enters the "good" conductor region when  $\omega$  gets sufficiently small.

wave vector;

$$k = \frac{\omega}{c} \sqrt{\mu \epsilon}$$

for a good conductor where  $\epsilon_2 \gg \epsilon_1$ ,

$$\epsilon \sim i\epsilon_2 = \frac{4\pi i\sigma_0}{\omega}$$

$$k = k_1 + ik_2 = \frac{\omega}{c} \sqrt{\mu \frac{4\pi i\sigma_0}{\omega}} \quad \sqrt{i} = \frac{1+i}{\sqrt{2}}$$

$$k_1 = k_2 = \frac{\omega}{c} \sqrt{\frac{4\pi\mu\sigma_0}{2\omega}} = \frac{1}{c} \sqrt{2\pi\mu\sigma_0\omega}$$

for  $\vec{k} = k\hat{z}$ ,

$$\vec{E} = \vec{E}_\omega e^{i(kz - \omega t)} = \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$$

$$\delta \equiv 1/k_2 = \frac{c}{\sqrt{2\pi\mu\sigma_0\omega}} \quad \text{"skin depth"}$$

distance wave propagates into conductor

$$\delta \sim 1/\sqrt{\omega} \quad \text{increases as } \omega \text{ decreases}$$

$\phi$  phase shift between oscillations of  $\vec{E}$  and  $\vec{H}$

$$\phi = \arctan(k_2/k_1) \approx \arctan(1) = 45^\circ$$

$$\text{Amplitude ratio } \frac{|\vec{H}_\omega|}{|\vec{E}_\omega|} = \frac{c|k|}{\omega\mu} = \frac{\sqrt{2}c}{\omega\mu} k_1$$

$$= \frac{\sqrt{2}c}{\omega\mu} \frac{1}{c} \sqrt{2\pi\mu\sigma_0\omega}$$

$$= \sqrt{\frac{4\pi\sigma_0}{\omega\mu}} \sim 1/\sqrt{\omega}$$

as  $\omega \rightarrow 0$ , most of the energy of the wave is carried by the magnetic field part

② high frequencies  $\omega \gg 1/\tau$ ,  $\omega \gg \omega_0$

$$\epsilon_b(\omega) \approx 1$$

$$\sigma(\omega) \approx \frac{\sigma_0}{-i\omega\tau} = \frac{ime^2\tau}{m\omega\tau} = \frac{ime^2}{m\omega}$$

pure imaginary  
indep of  $\tau$

$$\epsilon(\omega) \approx 1 + \frac{4\pi i\sigma}{\omega} \approx 1 - \frac{4\pi me^2}{m\omega^2}$$

$$\boxed{\epsilon(\omega) \approx 1 - \frac{\omega_p^2}{\omega^2}}$$

$$\omega_p \equiv \sqrt{\frac{4\pi me^2}{m}}$$

plasma freq of the  
conduction electrons

$\epsilon(\omega)$  is real

1) If  $\omega > \omega_p$  then  $\epsilon > 0$

$\Rightarrow$  transparent propagation

$$k = k_1 = \frac{\omega}{c} \sqrt{\mu\epsilon} \text{ is pure real}$$

$$k_2 \approx 0$$

2) If  $\omega < \omega_p$  then  $\epsilon < 0$

$\Rightarrow$  total reflection  $k \approx ik_2$

$$k_1 \approx 0$$

$$k_2 = \frac{\omega}{c} \sqrt{\mu|\epsilon|}$$

$k$  is pure imaginary

$\omega_p$  gives cross over between total reflection  
and transparent propagation

for typical metals

$$\tau \sim 10^{-14} \text{ sec}$$

$$\omega_p \sim 10^{16} \text{ sec}^{-1}$$

$$\lambda_p = \frac{2\pi c}{\omega_p} \sim 3 \times 10^3 \text{ \AA} \quad (\text{visible is } \lambda \sim 5 \times 10^3 \text{ \AA})$$

Example: The ionosphere is a layer of charged gas surrounding the earth.

In many respects the charged particles of the ionosphere behave like conduction electrons in a metal. The plasma freq of the ionosphere is such that

for AM radio  $\omega_{AM} < \omega_p \Rightarrow$  AM radio signals reflected back to earth

for FM radio  $\omega_{FM} > \omega_p \Rightarrow$  FM radio signals propagate through ionosphere into space

Explains why you can pick up AM stations from far away - they get reflected back

But you can only pick up local FM stations.

## Longitudinal modes in conductors

ie  $\vec{H}_\omega$  or  $\vec{E}_\omega$  not  $\perp \vec{k}$   
magnetic field

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow i\mu \vec{k} \cdot \vec{H}_\omega = 0 \Rightarrow \vec{H}_\omega \perp \vec{k} \text{ transverse}$$

or  $\vec{k} = 0$  spatially uniform  $\vec{H}$

if  $\vec{k} = 0$  then Faraday

$$i\vec{k} \times \vec{E}_\omega = i\omega\mu \vec{H}_\omega = 0 \Rightarrow \omega = 0$$

" as  $\vec{k} = 0$

So only possible longitudinal  $\vec{H}$  is spatially uniform, constant in time.

## electric field

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho_f \Rightarrow \epsilon \epsilon(\omega) \vec{k} \cdot \vec{E}_\omega = 0 \Rightarrow \vec{E}_\omega \perp \vec{k} \text{ transverse}$$

or  $\epsilon(\omega) = 0$

If  $\vec{E}_\omega \parallel \vec{k}$  but  $\epsilon(\omega) = 0$ , then can satisfy all other Maxwell equations.

$$i\vec{k} \times \vec{E}_\omega = \frac{i\omega\mu}{c} \vec{H}_\omega \Rightarrow \vec{H}_\omega = 0$$

$$\Rightarrow i\rho_0 \vec{k} \cdot \vec{H}_\omega = 0 \quad \text{and} \quad i\vec{k} \times \vec{H}_\omega = -\frac{i\omega\epsilon(\omega)}{c} \vec{E}_\omega$$

" as  $\vec{H}_\omega = 0$       " as  $\epsilon(\omega) = 0$

So we can have longitudinal electric field oscillation when  $\epsilon(\omega) = 0$



low freq  $\omega \ll \omega_0$   $\omega \tau \ll 1$

$$\epsilon \approx \epsilon_b(\omega) + \frac{4\pi i \sigma_0}{\omega}$$

$$\epsilon(\omega) = 0 \text{ when } \omega = -\frac{4\pi \sigma_0}{\epsilon_b(\omega)}$$

$$\vec{E}(\vec{r}, t) = \vec{E}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \vec{E}_\omega e^{-\frac{4\pi \sigma_0}{\epsilon_b(\omega)} t} e^{i\vec{k} \cdot \vec{r}}$$

If set up a longitudinal  $\vec{E}$  field, it decays to zero exponentially with ~~time const~~ decay time  $\epsilon_b(\omega)/4\pi\sigma_0$ . This is consistent with assumption the  $\vec{E} = 0$  inside a conductor for electrostatics

in statics  $\vec{E} = -\vec{\nabla}\phi \Rightarrow \vec{E} \sim -ik\phi_k e^{i\vec{k} \cdot \vec{r}}$  is longitudinal

high freq  $\omega \gg 1/\tau$ ,  $\omega \gg \omega_0$

$$\epsilon(\omega) \approx 1 + \frac{4\pi C \sigma_0}{\omega} = 1 - \frac{\omega_p^2}{\omega^2} \quad \omega_p^2 = \frac{4\pi m e^2}{m}$$

$$\epsilon = 0 \text{ when } \omega = \omega_p$$

So we have oscillatory longitudinal  $\vec{E}$  only when  $\omega = \omega_p$ , independent of  $\vec{k}$ .

$$\vec{E} = \vec{E}_\omega e^{i\vec{k} \cdot \vec{r}} e^{-i\omega_p t}$$

This is called a plasma oscillation. When one quantizes this oscillatory mode, it is called a plasmon.

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho \Rightarrow \rho = \frac{i\vec{k} \cdot \vec{E}_\omega}{4\pi} e^{i\vec{k} \cdot \vec{r}} e^{-i\omega_p t} \left[ \begin{array}{l} \text{plasma osc.} \\ \text{is a charge} \\ \text{density oscillation} \end{array} \right]$$