$\Rightarrow$ $\theta_2'' = 0$ $\quad$ attenuation factor for the transmitted
$\vec{k}_2'' = k_2''\,\hat{z}$ $\quad$ wave is $e^{-k_2 z}$

$\Rightarrow$ planes of constant amplitude are always parallel to the interface no matter what the angle of incidence θ_0

Having found θ_2'' there are still three quantities we must yet find in order to characterize the transmitted wave. These are θ_2', k_2', k_2''.

To solve for these we will need 3 equations

one is: $\quad k_0 \sin\theta_0 = k_2' \sin\theta_2'$ $\quad$ (1)
(from boundary condition)

where $k_0 = \frac{\omega}{c}\sqrt{\mu_a \epsilon_a} = \frac{\omega}{c} n_a$ $\quad$ dispersion relation in medium a

The other two come from equating the real and imaginary parts of the dispersion relation in medium b.

$$k_2^2 = \frac{\omega^2}{c^2}\mu_b \epsilon_b = \frac{\omega^2}{c^2}\mu_b(\epsilon_{b1} + i\epsilon_{b2})$$

$$\begin{aligned} k_2^2 &= (\vec{k}_2' + i\vec{k}_2'')\cdot(\vec{k}_2' + i\vec{k}_2'') \\ &= (k_2')^2 - (k_2'')^2 + 2i\,\vec{k}_2'\cdot\vec{k}_2'' \\ &= (k_2')^2 - (k_2'')^2 + 2i\,k_2' k_2'' \cos\theta_2' \end{aligned}$$

equate real and imaginary parts

$$(k_2')^2 - (k_2'')^2 = \frac{\omega^2}{c^2}\mu_b \varepsilon_{b1} \qquad ②$$

$$2k_2' k_2'' \cos\theta_2' = \frac{\omega^2}{c^2}\mu_b \varepsilon_{b2} \qquad ③$$

Use ② and ③ to solve for k_2' and k_2'' in terms of θ_2'

$$② \Rightarrow (k_2')^2 = (k_2'')^2 + \frac{\omega^2}{c^2}\mu_b \varepsilon_{b1} \qquad ④$$

$$③ \Rightarrow k_2'' = \frac{\omega^2}{c^2}\frac{\mu_b \varepsilon_{b2}}{2k_2' \cos\theta_2'} \qquad ⑤$$

plug ⑤ into ④

$$(k_2')^2 = \left(\frac{\omega^2}{c^2}\frac{\mu_b \varepsilon_{b2}}{2k_2'\cos\theta_2'}\right)^2 + \frac{\omega^2}{c^2}\mu_b \varepsilon_{b1}$$

$$\Rightarrow (k_2')^4 - \frac{\omega^2}{c^2}\mu_b \varepsilon_{b1}(k_2')^2 - \frac{\omega^4}{c^4}\frac{\mu_b^2 \varepsilon_{b2}^2}{4\cos^2\theta_2'} = 0$$

solve quadratic formula

$$(k_2')^2 = \frac{\omega^2 \mu_b \varepsilon_{b1}}{2c^2} + \sqrt{\frac{\omega^4 \mu_b^2 \varepsilon_{b1}^2}{4c} + \frac{\omega^4 \mu_b^2 \varepsilon_{b2}^2}{4c^4\cos^2\theta_2'}}$$

take (+) solution only since $(k_2')^2$ must be positive

$$= \frac{\omega^2 \mu_b}{c^2}\left[\frac{\varepsilon_{b1}}{2} + \frac{1}{2}\sqrt{\varepsilon_{b1}^2 + \frac{\varepsilon_{b2}^2}{\cos^2\theta_2'}}\right]$$

⑥ $$\boxed{k_2' = \frac{\omega}{c}\sqrt{\mu_b}\left[\frac{1}{2}\epsilon_{b1} + \frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_2'}}\right]^{1/2}}$$

then get k_2'' from ④

$$(k_2'')^2 = (k_2')^2 - \frac{\omega^2}{c^2}\mu_b\varepsilon_{b1}$$

⑦ $$\boxed{k_2'' = \frac{\omega}{c}\sqrt{\mu_b}\left[-\frac{1}{2}\epsilon_{b1} + \frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_2'}}\right]^{1/2}}$$

Note, these reduce to what we found earlier for the real and imaginary parts of the wave vector for a plane wave in a medium with complex ε, IF we take $\theta_2' = 0$. We will have $\theta_2' = 0$ for normal incidence $\theta_0 = 0$.

Both k_2' and k_2'' above still depend on the angle of refraction θ_2'. We can close the set of equations by adding in Eq ①

$$k_0 \sin\theta_0 = k_2' \sin\theta_2'$$

⑧ or $$\boxed{\frac{\omega}{c} n_a \sin\theta_0 = k_2' \sin\theta_2'}$$

where $n_a = \frac{k_0 c}{\omega} = \sqrt{\mu_a \varepsilon_a}$

Since the pair of equations ⑥ and ⑧ only involve the unknowns k_2' and θ_2' we can use them to eliminate k_2' and get a final single equation that determines θ_2')

Define index of refraction in medium b

$$n_b = \sqrt{\mu_b \varepsilon_{b1}}$$

Then

$$\frac{\omega}{c} n_a \sin\theta_0 = \frac{\omega}{c} n_b \left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\varepsilon_{b2}^2}{\varepsilon_{b1}^2 \cos^2\theta_2'}}\right]^{1/2} \sin\theta_2'$$

or

$$\boxed{n_a \sin\theta_0 = n_b \sin\theta_2' \left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\varepsilon_{b2}^2}{\varepsilon_{b1}^2 \cos^2\theta_2'}}\right]^{1/2}}$$

This is the analog of Snell's law for propagation into a medium with complex dielectric function ε

Cases

① For a nearly transparent material with $\varepsilon_{b2} \ll \varepsilon_{b1}$ we can expand in $\frac{\varepsilon_{b2}}{\varepsilon_{b1}}$ to get

$$n_a \sin\theta_0 = n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{4\epsilon_{b1}^2 \cos^2\theta_2'} \right]^{1/2}$$

$$\cong n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2 \cos^2\theta_2'} \right]$$

small correction to Snell's law

for $\frac{\epsilon_{b2}}{\epsilon_{b1}} \ll 1$ can solve iteratively

to lowest order: $n_a \sin\theta_0 \simeq n_b \sin\theta_2'$

$$\Rightarrow \cos^2\theta_2' = 1 - \sin^2\theta_2' = 1 - \left(\frac{n_a}{n_b}\sin\theta_0\right)^2$$

so to next order

$$n_a \sin\theta_0 \simeq n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2 \left(1 - \frac{n_a^2}{n_b^2}\sin^2\theta_0\right)} \right]$$

or $$\sin\theta_2' \simeq \frac{n_a}{n_b}\sin\theta_0 \frac{1}{\left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2 \left(1 - \frac{n_a^2}{n_b^2}\sin^2\theta_0\right)} \right]}$$

$$\leq \frac{n_a}{n_b}\sin\theta_0$$

result is that θ_2' is <u>smaller</u> than Snell's law would predict.

(2) for a good conductor, or absorbing region of a dielectric, $\epsilon_{b2} \gg \epsilon_{b1}$

to lowest order

$$n_a \sin\theta_0 = \sqrt{\mu_b \epsilon_{b1}} \left[\frac{1}{2} \frac{\epsilon_{b2}}{\epsilon_{b1} \cos\theta_2'} \right]^{1/2} \sin\theta_2'$$

$$n_a \sin\theta_0 = \sqrt{\frac{\mu_b \epsilon_{b2}}{2}} \frac{\sin\theta_2'}{\sqrt{\cos\theta_2'}}$$

very different from Snell's Law!

Snell's Law only holds if both media are transparent

$$\Rightarrow n_a^2 \sin^2\theta_0 = \frac{\mu_b \epsilon_{b2}}{2} \frac{\sin^2\theta_2'}{\cos\theta_2'} = \frac{\mu_b \epsilon_{b2}}{2} \frac{1-\cos^2\theta_2'}{\cos\theta_2'}$$

$$\Rightarrow \cos^2\theta_2' + \left(\frac{2}{\mu_b \epsilon_{b2}}\right)(n_a^2 \sin^2\theta_0) \cos\theta_2' - 1 = 0$$

solve quadratic equation in $\cos\theta_2'$ to determine $\cos\theta_2'$. Then can use that in expressions for k_2' and k_2'' to determine those. We will have in the $\epsilon_{b2} \gg \epsilon_{b1}$ case that $k_1' \approx k_2'$

Reflection coefficients

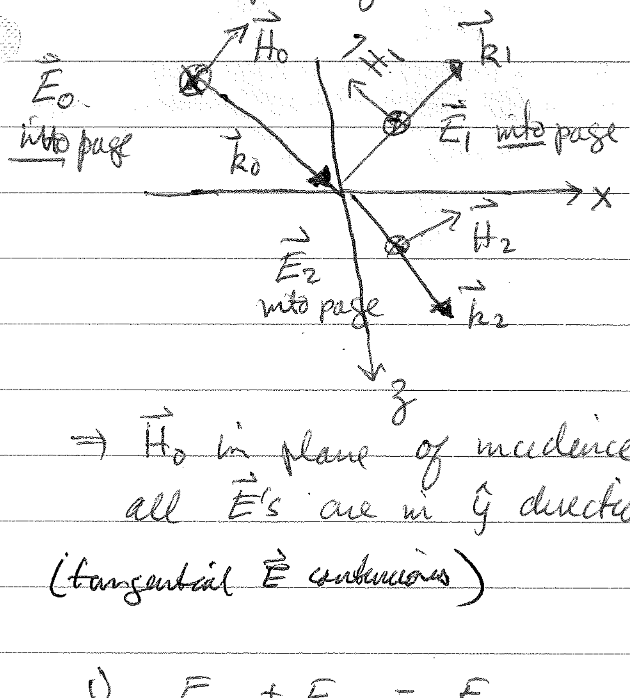
Now we compute the amplitude of the reflected wave to determine how much of incident wave is reflected and how much is transmitted.

Consider two cases (1) $\vec E_\omega$ is $\perp$ plane of incidence

(2) $\vec E_\omega$ lies in the plane of incidence

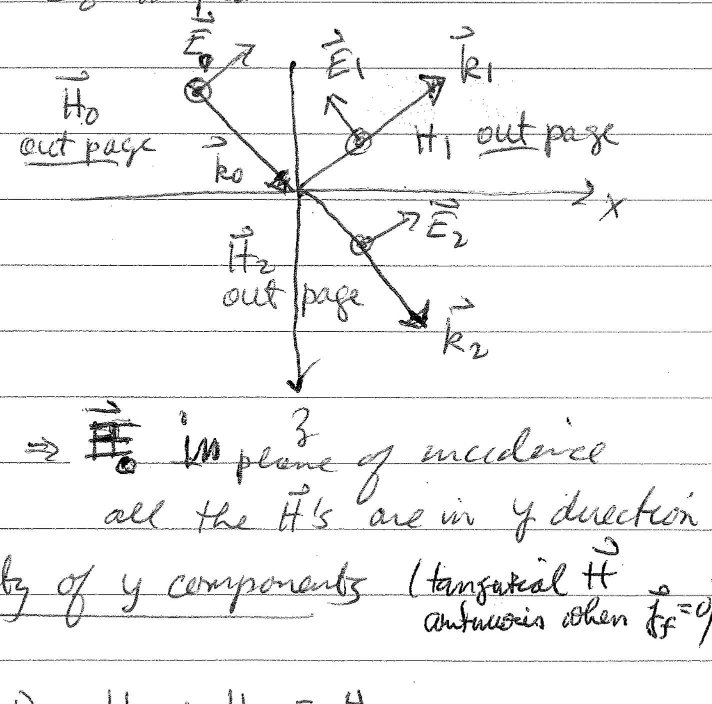
"plane of incidence" is the plane spanned by the wave vector $\vec k_0$ and the normal to the interface – in our case it is the xz plane

(1) $\vec E_0 \perp$ plane of incidence



$\Rightarrow \vec H_0$ in plane of incidence
all $\vec E$'s are in $\hat y$ direction

(2) $\vec E_0 \parallel$ plane incidence



$\Rightarrow \vec E_0$ in plane of incidence
all the $\vec H$'s are in y direction

(tangential $\vec E$ continuous) Continuity of y components (tangential $\vec H$ continuous when $\vec J_f=0$)

1) $E_0 + E_1 = E_2$ | 1) $H_0 + H_1 = H_2$

(tangential $\vec H$ continuous when $\vec J_f = 0$) Continuity of x components (tangential $\vec E$ continuous)

$H_{0x} + H_{1x} = H_{2x}$ | $E_{0x} + E_{1x} = E_{2x}$

Faraday

$\frac{i\mu\omega}{c}\vec H = i\vec k\times\vec E \Rightarrow H_x = \frac{k_z c}{\omega\mu}E_y$

Ampere

$-\frac{i\omega\epsilon}{c}\vec E = i\vec k\times\vec H \Rightarrow E_x = \frac{-k_z c}{\omega\epsilon}H_y$

$\Rightarrow$

2) $\frac{k_{0z}}{\mu_a}(E_0 - E_1) = \frac{k_{2z}}{\mu_b} E_2$

solve (1) and (2) for E_1 and E_2 in terms of E_0

$$E_1 = \frac{\mu_b k_{0z} - \mu_a k_{2z}}{\mu_b k_{0z} + \mu_a k_{2z}} E_0$$

$$E_2 = \frac{2\mu_b k_{0z}}{\mu_a k_{2z} + \mu_b k_{0z}} E_0$$

2) $\frac{k_{0z}}{\epsilon_a}(H_0 - H_1) = \frac{k_{2z}}{\epsilon_b} H_2$

solve (1) and (2) for H_1 and H_2 in terms of H_0

$$H_1 = \frac{\epsilon_b k_{0z} - \epsilon_a k_{2z}}{\epsilon_b k_{0z} + \epsilon_a k_{2z}} H_0$$

$$H_2 = \frac{2\epsilon_b k_{0z}}{\epsilon_a k_{2z} + \epsilon_b k_{0z}} H_0$$

Define reflection coefficient in terms of the transported energy $R = \frac{|E_1|^2}{|E_0|^2} = \frac{|H_1|^2}{|H_0|^2} = \frac{\text{energy reflected current}}{\text{incident energy current}}$

<u>Reflection coefficients</u>

① $\vec{E}_0 \perp$ plane incidence

$$R_\perp = \frac{|E_1|^2}{|E_0|^2} = \left| \frac{\mu_b k_{0z} - \mu_a k_{2z}}{\mu_b k_{0z} + \mu_a k_{2z}} \right|^2$$

② $\vec{E}_0 \parallel$ plane incidence

$$R_\parallel = \frac{|H_1|^2}{|H_0|^2} = \left| \frac{\epsilon_b k_{0z} - \epsilon_a k_{2z}}{\epsilon_b k_{0z} + \epsilon_a k_{2z}} \right|^2$$

Note: above are correct for an <u>arbitrary</u> medium B

i) Consider region of "total reflection"

$\Rightarrow$ $\text{Im}\,\epsilon_b = \epsilon_{b2} \approx 0$
$\text{Re}\,\epsilon_b = \epsilon_{b1} < 0$ $\Big\} \Rightarrow \vec{k}_2 = i\vec{K}_2$ where $\vec{K}_2$ is real

ie $\vec{k}_2$ pure imaginary

$$\Rightarrow \quad R_\perp = \left| \frac{\mu_b k_{0z} - i\mu_a K_{2z}}{\mu_b k_{0z} + i\mu_a K_{2z}} \right|^2$$

$$R_\parallel = \left| \frac{\epsilon_b k_{0z} - i\epsilon_a K_{2z}}{\epsilon_b k_{0z} + i\epsilon_a K_{2z}} \right|^2$$

both are of the form $\left| \frac{a - ib}{a + ib} \right|^2 = 1$ when a, b real

$\Rightarrow R_\perp = R_\parallel = 1$

confirms that the material is completely reflecting

ii) Next consider when medium B is transparent

ϵ_b is real and $\epsilon_b > 0$

$$k_{0z} = \frac{\omega}{c}\sqrt{\mu_a \epsilon_a}\cos\theta_0 = \frac{\omega}{c} n_a \cos\theta_0$$
$$k_{2z} = \frac{\omega}{c}\sqrt{\mu_b \epsilon_b}\cos\theta_2 = \frac{\omega}{c} n_b \cos\theta_2$$

Snell's law holds so $n_a \sin\theta_0 = n_b \sin\theta_2$

can write $R_\perp$ and $R_\parallel$ as functions of θ_0

for simplicity take $\mu_a = \mu_b = 1$

① $$R_\perp = \left(\frac{n_a \cos\theta_0 - n_b \cos\theta_2}{n_a \cos\theta_0 + n_b \cos\theta_2}\right)^2 = \left(\frac{\cos\theta_0 - \left(\frac{\sin\theta_0}{\sin\theta_2}\right)\cos\theta_2}{\cos\theta_0 + \left(\frac{\sin\theta_0}{\sin\theta_2}\right)\cos\theta_2}\right)^2$$

$$= \left(\frac{\sin\theta_2 \cos\theta_0 - \sin\theta_0 \cos\theta_2}{\sin\theta_2 \cos\theta_0 + \sin\theta_0 \cos\theta_2}\right)^2$$

$$R_\perp = \left(\frac{\sin(\theta_0 - \theta_2)}{\sin(\theta_0 + \theta_2)}\right)^2$$

for $\theta_0 = 0$, ie normal incidence, $\theta_2 = 0$, plug into ①

$$\Rightarrow R_\perp = \left(\frac{n_a - n_b}{n_a + n_b}\right)^2$$ if $n_a = n_b$, no reflection! (not surprising!)

② $$R_\parallel = \left(\frac{\epsilon_b n_a \cos\theta_0 - \epsilon_a n_b \cos\theta_2}{\epsilon_b n_a \cos\theta_0 + \epsilon_a n_b \cos\theta_2}\right)^2$$ use $\sqrt{\epsilon_b} = n_b$, $\sqrt{\epsilon_a} = n_a$

$$= \left(\frac{n_b \cos\theta_0 - n_a \cos\theta_2}{n_b \cos\theta_0 + n_a \cos\theta_2}\right)^2$$

$$= \left(\frac{\cos\theta_0 - \left(\frac{\sin\theta_2}{\sin\theta_0}\right)\cos\theta_2}{\cos\theta_0 + \left(\frac{\sin\theta_2}{\sin\theta_0}\right)\cos\theta_2}\right)^2$$

$$= \left(\frac{\sin\theta_0 \cos\theta_0 - \sin\theta_2 \cos\theta_2}{\sin\theta_0 \cos\theta_0 + \sin\theta_2 \cos\theta_2}\right)^2$$

$$R_\parallel = \left(\frac{\tan(\theta_0 - \theta_2)}{\tan(\theta_0 + \theta_2)}\right)^2$$ ← after some algebra!

for $\theta_0 = 0$, then $\theta_2 = 0$, plug into ②

$$R_{\parallel} = \left(\frac{\epsilon_b n_a - \epsilon_a n_b}{\epsilon_b n_a + \epsilon_a n_b}\right)^2 = \left(\frac{n_b - n_a}{n_b + n_a}\right)^2 \quad \text{same as } R_{\perp}$$

So for $\theta_0 = 0$, $R_{\parallel} = R_{\perp}$ — this must be so since for $\theta_0 = 0$ there is no distinction between the $\perp$ and $\parallel$ cases

If $n_b = n_a$, $R_{\perp} = R_{\parallel} = 0$ no reflective wave

When $\theta_0 + \theta_2 = \pi/2$, then $\tan(\theta_0 + \theta_2) \to \infty$ and $R_{\parallel} = 0$

This occurs at an angle of incidence known as Brewster's angle θ_B, determined by

$$\underbrace{n_a \sin \theta_B}_{\theta_0} = n_b \sin\Big(\underbrace{\tfrac{\pi}{2} - \theta_B}_{\theta_2}\Big) = n_b \cos \theta_B$$

$$\Rightarrow \boxed{\tan \theta_B = \frac{n_b}{n_a}}$$

For incident wave at θ_B, reflected wave always has $\vec{E}_1 \perp$ plane of incidence, since $R_{\parallel} = 0$. If incoming wave has $\vec{E}_0 \parallel$ plane of incidence, then it gets completely transmitted. If $\vec{E}_0$ in general direction, reflected wave is always linearly polarized with $\vec{E}_1 \perp$ plane of incidence. — This is one method to create polarized light wave.