Unit 1-3: Faraday's Law, Maxwell's Correction to Ampere's Law, Systems of Units

So far we have,

$$\text{electrostatics:} \qquad \boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi k_1\,\rho\,, \qquad \boldsymbol{\nabla}\times\mathbf{E} = 0 \tag{1.3.1}$$

$$\text{magnetostatics:} \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0\,, \qquad \boldsymbol{\nabla}\times\mathbf{B} = 4\pi k_2\,\mathbf{j} \tag{1.3.2}$$

$$\text{charge conservation:} \qquad \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0 \tag{1.3.3}$$

In statics, electric fields $\mathbf{E}$ and magnetic fields $\mathbf{B}$ are decoupled from each other. In statics, each term in the charge conservation equation is separately equal to zero. We now want to consider non-static situations, where ρ and $\mathbf{j}$ are coupled through the charge conservation equation, and we will find that $\mathbf{E}$ and $\mathbf{B}$ are also coupled.

Faraday's Law of Induction

When magnetic fields change in time, it was experimentally determined that one of the key equations of electrostatics, $\boldsymbol{\nabla}\times\mathbf{E} = 0$, no longer holds true. Instead we have Faraday's Law of induction. For a loop C bounding the surface S,

$$\oint_C \boldsymbol{d\ell}\cdot\mathbf{E} = -k_4\frac{d}{dt}\left[\int_S da\,\hat{\mathbf{n}}\cdot\mathbf{B}\right] \tag{1.3.4}$$

where k_4 is another new universal constant of nature. The integral around the curve C on the left hand side of the above is called the electromotive force, denoted as emf or $\mathcal{E}$. When C is a physical wire loop, the emf can be viewed as the voltage drop around. The integral on the right hand side is the flux of the magnetic field through the surface S, and is denoted as Φ. So Faraday's Law can also be written as

$$\mathcal{E} = -k_4\frac{d\Phi}{dt} \qquad \text{the voltage drop around a closed loop} \sim \text{(-) time rate of change of the magnetic flux through the loop} \tag{1.3.5}$$

Note, in applying Faraday's Law, when computing the flux Φ, the outward normal $\hat{\mathbf{n}}$ to the surface S must be taken in the direction consistent with the direction one goes around the loop in computing the emf $\mathcal{E}$ via the "right hand rule." If one curls the fingers of one's right hand in the direction $\boldsymbol{d\ell}$ going around the loop, then one's thumb points in the direction of $\hat{\mathbf{n}}$.

Using Stoke's Theorem of vector calculus we can write

$$\oint_C \boldsymbol{d\ell}\cdot\mathbf{E} = \int_S da\,\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\mathbf{E} = -k_4\int_S da\,\hat{\mathbf{n}}\cdot\frac{\partial\mathbf{B}}{\partial t} \tag{1.3.6}$$

Since the above must be true for any surface S, it follows that the integrands of the last two terms must be equal. We thus get Faraday's Law in differential form,

$$\boxed{\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r},t) = -k_4\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}} \tag{1.3.7}$$

We see that Faraday's Law gives a coupling between the electric and magnetic fields. This coupling goes away in static situations where $\partial\mathbf{B}/\partial t = 0$.

Maxwell's Correction to Ampere's Law

In our derivation of $\boldsymbol{\nabla}\times\mathbf{B} = 4\pi k_2\,\mathbf{j}$ we used the magnetostatic condition $\boldsymbol{\nabla}\cdot\mathbf{j} = 0$. But that is only true in magnetostatic situations, it is not true for general time dependent situations. This led Maxwell to realize that Ampere's Law must have a correction when applied to non-static situations. Alternatively, if Ampere's Law remained true in general, $\boldsymbol{\nabla}\times\mathbf{B} = 4\pi k_2\,\mathbf{j}$, then taking the divergence of both sides would give,

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) = 4\pi k_2\boldsymbol{\nabla}\cdot\mathbf{j} \tag{1.3.8}$$

But one must have $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{B})=0$ for *any* vector function $\mathbf{B}(\mathbf{r})$ (divergence of a curl always vanishes). So the left hand side of the above is always zero, which would imply that the right hand side must also be zero, or $\boldsymbol{\nabla}\cdot\mathbf{j}=0$. But we know that $\boldsymbol{\nabla}\cdot\mathbf{j}\neq 0$ in general, it is only true in magnetostatics. More generally we know from charge conservation that $\boldsymbol{\nabla}\cdot\mathbf{j}=-\partial\rho/\partial t$. So Ampere's Law as we know it so far cannot remain correct once we have time varying situations where $\partial\rho/\partial t\neq 0$.

Maxwell therefore proposed a correction to Ampere's Law,

$$\boldsymbol{\nabla}\times\mathbf{B}=4\pi k_2\,\mathbf{j}+\mathbf{W} \tag{1.3.9}$$

where $\mathbf{W}$ must be chose so that the law is consistent with charge conservation. Taking the divergence of both sides of the above then gives,

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{B})=0=4\pi k_2\,\boldsymbol{\nabla}\cdot\mathbf{j}+\boldsymbol{\nabla}\cdot\mathbf{W}\quad\Rightarrow\quad\boldsymbol{\nabla}\cdot\mathbf{W}=-4\pi k_2\boldsymbol{\nabla}\cdot\mathbf{j}=4\pi k_2\frac{\partial\rho}{\partial t} \tag{1.3.10}$$

In the last step we used the law of conservation of charge. Now we use Gauss' Law of electrostatics $\boldsymbol{\nabla}\cdot\mathbf{E}=4\pi k_1\,\rho$ to write,

$$\boldsymbol{\nabla}\cdot\mathbf{W}=4\pi k_2\frac{\partial}{\partial t}\left(\frac{\boldsymbol{\nabla}\cdot\mathbf{E}}{4\pi k_1}\right)=\boldsymbol{\nabla}\cdot\left(\frac{k_2}{k_1}\frac{\partial\mathbf{E}}{\partial t}\right) \tag{1.3.11}$$

The simplest guess is then that

$$\mathbf{W}=\frac{k_2}{k_1}\frac{\partial\mathbf{E}}{\partial t} \tag{1.3.12}$$

Note, this is not the only possible solution to Eq. (1.3.11). A more general solution would be $\mathbf{W}=(k_2/k_2)(\partial\mathbf{E}/\partial t)+\boldsymbol{\nabla}\times\mathbf{A}$ for any vector function $\mathbf{A}$. However the simple guess of Eq. (1.3.12) turns out to be correct. We therefore have Ampere's Law with Maxwell's correction,

$$\boxed{\boldsymbol{\nabla}\times\mathbf{B}=4\pi k_2\,\mathbf{j}+\frac{k_2}{k_1}\frac{\partial\mathbf{E}}{\partial t}} \tag{1.3.13}$$

So just like Faraday's Law, Maxwell's correction to Ampere's Law gives a coupling between the electric and magnetic fields. But this coupling goes away in static situations were $\partial\mathbf{E}/\partial t=0$.

Electromagnetic Waves

With Faraday's Law and the corrected Ampere's Law, we now find that Maxwell's equations allow waves as a solution. Use the vector identity $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{B})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{B})-\nabla^2\mathbf{B}$ which is true for any vector field $\mathbf{B}$. Now if $\mathbf{B}$ is the magnetic field we know that $\boldsymbol{\nabla}\cdot\mathbf{B}=0$, so this becomes $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{B})=-\nabla^2\mathbf{B}$. Assume there are no charge or current sources, $\rho=0$ and $\mathbf{j}=0$. Now take the curl of Ampere's Law,

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{B})=-\nabla^2\mathbf{B}=\boldsymbol{\nabla}\times\left(\frac{k_2}{k_2}\frac{\partial\mathbf{E}}{\partial t}\right)=\frac{k_2}{k_1}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{E})=-\frac{k_4k_2}{k_1}\frac{\partial^2\mathbf{B}}{\partial t^2} \tag{1.3.14}$$

where in the last step we used Faraday's Law, $\boldsymbol{\nabla}\times\mathbf{E}=-k_4\partial\mathbf{B}/\partial t$. This then can be written as,

$$\nabla^2\mathbf{B}-\frac{k_4k_2}{k_1}\frac{\partial^2\mathbf{B}}{\partial t^2}=0 \tag{1.3.15}$$

This is just the wave equation! The combination k_1/k_4k_2 has the units of $(\text{velocity})^2$, and gives the velocity of propagation of the wave. Experimentally it was then determined that

$$\frac{k_1}{k_4k_2}=c^2 \tag{1.3.16}$$

where c is the velocity of light in a vacuum. This experimental result, relating the velocity of light as measured in optical experiments, to the constants k_1, k_2, and k_4 measured in electromagnetic experiments, led to the conclusion that light is an electromagnetic wave!

Earlier in Eq. (1.2.5) we said that experimentally it was determined that $k_1/k_2k_3 = c^2$ (recall, k_3 is the constant that enters the Lorentz force). We thus conclude that $k_3 = k_4$.

Systems of Units

Recall, we said that the constants k_1 and k_2 are arbitrary – their values are determined by how we choose to define our units of charge and magnetic field. However, once having chose k_1 and k_2, the values of k_3 and k_4 are then fixed by, $k_3 = k_4$ and $k_1/k_4k_2 = c^2$.

Some common systems of electromagnetic units are then as given in the table below.

System	k_1	k_2	$k_3 = k_4$
MKS or SI	$\dfrac{1}{4\pi\epsilon_0}$	$\dfrac{\mu_0}{4\pi}$	1
Gaussian or CGS	1	$\dfrac{1}{c}$	$\dfrac{1}{c}$
Rationalized Gaussian	$\dfrac{1}{4\pi}$	$\dfrac{1}{4\pi\, c}$	$\dfrac{1}{c}$

In MKS units ϵ_0 is called the permittivity of free space, and μ_0 is called the permeability of free space. The condition $k_1/k_4k_2 = c^2$ requires $\epsilon_0\mu_0 = 1/c^2$. MKS units is often used in engineering. CGS units are usually favored in physics.

In MKS units, charges are measured in "coulombs," current is measured in "amperes," and magnetic field is measured in "tesla=weber/m^2." These are the historical units.

In CGS units, charges are measured in "statcoulombs," current is measured in "statamperes," and magnetic field is measured in "gauss" where 1 tesla = 10^4 gauss.

In this class we will be using CGS units. In CGS units, Maxwell's equations are,

$$1)\quad \boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\,\rho \qquad\qquad 3)\quad \boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{1.3.17}$$

$$2)\quad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \qquad\qquad 4)\quad \boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \tag{1.3.18}$$

and the Lorentz force is

$$\mathbf{F} = q\mathbf{E} + q\frac{\mathbf{v}}{c}\times\mathbf{B} \tag{1.3.19}$$

Equations (2) and (3) are referred to as the *homogeneous* Maxwell's equations – they do not involve the sources ρ and $\mathbf{j}$. Equations (1) and (4) are referred to as the *inhomogeneous* Maxwell's equations – they do involve the sources ρ and $\mathbf{j}$.

(1) Gauss' Law for the electric field says that charge is the source of the $\mathbf{E}$ field. Field lines of $\mathbf{E}$ begin and end at point charges.

(2) Faraday's Law of induction says that time varying magnetic fields produce a circulating $\mathbf{E}$ field.

(3) Gauss' Law for the magnetic field says there are no magnetic monopoles. Field lines of $\mathbf{B}$ are continuous; they either close upon themselves or go off to infinity, they cannot begin nor end at any point.

(4) Ampere's Law + Maxwell's correction says that electric current, and a time varying $\mathbf{E}$, is a source for a circulating $\mathbf{B}$ field. Maxwell's correction is necessary to have charge conservation and to give electromagnetic waves.