Unit 2-5: Magnetostatics

We have Maxwell's equations for magnetostatics:

$$\boldsymbol{\nabla}\cdot\mathbf{B}=0 \qquad \text{and} \qquad \boldsymbol{\nabla}\times\mathbf{B}=\frac{4\pi}{c}\mathbf{j} \quad \text{Ampere's Law for } \textit{magnetostatics} \tag{2.5.1}$$

In terms of the magnetic vector potential $\mathbf{A}$ we have $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$ and so Ampere's Law can be written as,

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})-\nabla^2\mathbf{A}=\frac{4\pi}{c}\mathbf{j} \tag{2.5.2}$$

In the Coulomb gauge $\boldsymbol{\nabla}\cdot\mathbf{A}=0$, this becomes,

$$-\nabla^2\mathbf{A}=\frac{4\pi}{c}\mathbf{j} \tag{2.5.3}$$

The operator ∇^2 is usually applied to a scalar function. When applied to a vector function we mean

$$\nabla^2\mathbf{A}=(\nabla^2 A_x)\hat{\mathbf{x}}+(\nabla^2 A_y)\hat{\mathbf{y}}+(\nabla^2 A_z)\hat{\mathbf{z}} \tag{2.5.4}$$

$\nabla^2\mathbf{A}$ only has a simple expression when using Cartesian coordinates, as above. One could try to express it, for example, in spherical coordinates,

$$\nabla^2\mathbf{A}=\nabla^2(A_r\hat{\mathbf{r}}+A_\theta\hat{\boldsymbol{\theta}}+A_\varphi\hat{\boldsymbol{\varphi}}) \tag{2.5.5}$$

One might be tempted to write the above as $(\nabla^2 A_r)\hat{\mathbf{r}}+(\nabla^2 A_\theta)\hat{\boldsymbol{\theta}}+(\nabla^2 A_\varphi)\hat{\boldsymbol{\varphi}}$, but that would be *wrong!* The reason is that, unlike Cartesian coordinates where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ have fixed directions, the spherical unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\varphi}}$ change direction depending on what position $\mathbf{r}$ one is at. So when one takes the derivatives in ∇^2, one also has to take the derivatives of the spherical unit vectors. In principle it can be done (for example, using $\hat{\mathbf{r}}=\sin\theta\cos\varphi\,\hat{\mathbf{x}}+\sin\theta\sin\varphi\,\hat{\mathbf{y}}+\cos\theta\,\hat{\mathbf{z}}$ would allow one to take derivatives of $\hat{\mathbf{r}}$ with respect to θ and φ), but the resulting expression is not simple. So we will stick with Cartesian coordinates for expressing $\nabla^2\mathbf{A}$.

In the Coulomb gauge in magnetostatics, we see from Eq. (2.5.3) that the vector potential $\mathbf{A}$ is the solution to Poisson's equation, with $\mathbf{j}$ as the source. So, aside from this involving vector quantities, it is in many respects similar to the electrostatic Poisson's equation for ϕ, and we can use many of the same techniques in trying to solve for $\mathbf{A}$. In particular, for a localized current distribution $\mathbf{j}$, where $\mathbf{j}\to 0$ as $|\mathbf{r}|\to\infty$, we have that $\mathbf{A}$ is given by the usual Coulomb integral,

$$\mathbf{A}(\mathbf{r})=\frac{1}{c}\int d^3r'\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \qquad \text{this is three equations for the three components } A_x,\ A_y \text{ and } A_z. \tag{2.5.6}$$

We can therefore treat this with a magnetic multipole expansion. This will lead us to the expression of the magnetic dipole moment.

Magnetic Multipole Expansion and the Magnetic Dipole Moment

We saw that, for the scalar potential ϕ, one could develop a general multipole expansion in terms of the spherical harmonics $Y_{\ell m}(\theta,\varphi)$. One can do a similar expansion for $\mathbf{A}$ but it involves *vector spherical harmonics* – see Jackson Chapter 9 if you are interested. We won't discuss this. Instead we will take a straightforward approach and stop after the magnetic dipole term.

For $r \gg r'$ we can approximate

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{[r^2-2\mathbf{r}'\cdot\mathbf{r}+r'^2]^{1/2}}=\frac{1}{r}\,\frac{1}{\left[1-\frac{2\mathbf{r}'\cdot\mathbf{r}}{r^2}+\left(\frac{r'}{r}\right)^2\right]^{1/2}} \tag{2.5.7}$$

Now do a Taylor series expansion in (r'/r). To lowest order, $1/[1-\epsilon]^{1/2}=1+\epsilon/2$. Taking $\epsilon\propto r'/r$, we get to lowest order,

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}\approx\frac{1}{r}\left[1+\frac{\mathbf{r}'\cdot\mathbf{r}}{r^2}+\dots\right]=\frac{1}{r}+\frac{\mathbf{r}'\cdot\mathbf{r}}{r^3}+\dots \tag{2.5.8}$$

Note, the term $(r'/r)^2$ in the denominator of Eq. (2.5.7) doesn't appear in the above since it is of order ϵ^2, and we are only expanding to linear order in ϵ. Inserting this into the Coulomb integral of Eq. (2.5.6) we get,

$$\mathbf{A}(\mathbf{r}) = \frac{1}{r}\int \frac{d^3r'}{c}\,\mathbf{j}(\mathbf{r}') + \frac{1}{r^3}\int \frac{d^3r'}{c}\,\mathbf{j}(\mathbf{r}')(\mathbf{r}'\cdot\mathbf{r}) + \dots \quad = \quad ① \quad + \quad ② \tag{2.5.9}$$

<u>Consider the first term</u> ① $\quad \int d^3r\,\mathbf{j}(\mathbf{r})$

If we denote the subscript $i = 1, 2, 3$ as labeling the x, y, z components of a vector, so for example $r_1 = x$, $r_2 = y$, $r_3 = z$, then for the i-th component of this integral we can write,

$$\int d^3r\, j_i(\mathbf{r}) = \sum_{j=1}^{3}\int d^3r\, j_j\left(\frac{\partial r_i}{\partial r_j}\right) \qquad \text{since} \quad \left(\frac{\partial r_i}{\partial r_j}\right) = \delta_{ij}. \quad \text{Now integrate by parts,} \tag{2.5.10}$$

$$= \sum_j \left[\oint_S da\, j_j r_i - \int d^3r\left(\frac{\partial j_j}{\partial r_j}\right) r_i\right] \quad \text{where } S \text{ is the surface enclosing the volume of integration.} \tag{2.5.11}$$

Now if the volume of integration is all of space, so that the surface $S \to \infty$, and $\mathbf{j}(\mathbf{r})$ is localized so that $\mathbf{j} \to 0$ as $|\mathbf{r}| \to \infty$, then the surface integral will vanish. Summing on the index j, the second term above can be written as

$$-\int d^3r\,(\boldsymbol{\nabla}\cdot\mathbf{j})\, r_i = 0 \qquad \text{since } \boldsymbol{\nabla}\cdot\mathbf{j} = 0 \text{ in magnetostatics.} \tag{2.5.12}$$

We conclude that $\int d^3r\,\mathbf{j}(\mathbf{r}) = 0$ and the term ① vanishes. Thus the leading "monopole" term in the magnetic multipole expansion vanishes.

<u>Consider the second term</u> ② $\quad \int d^3r'\mathbf{j}(\mathbf{r}')(\mathbf{r}'\cdot\mathbf{r}) = \left[\int d^3r'\,\mathbf{j}(\mathbf{r}')\mathbf{r}'\right]\cdot\mathbf{r}$

Consider the ij-th element of the tensor $\int d^3r\,\mathbf{j}\,\mathbf{r}$.

$$\int d^3r\, j_i\, r_j = \sum_k \int d^3r\, j_k\, r_j\left(\frac{\partial r_i}{\partial r_k}\right) \qquad \text{then integrate by parts to get} \tag{2.5.13}$$

$$= \sum_k\left[\oint_S da\;\, j_k\, r_j\, r_i \;-\; \int d^3r\left(\frac{\partial[j_k\, r_j]}{\partial r_k}\right) r_i\right] \tag{2.5.14}$$

The surface integral vanishes as before when we take $S \to \infty$. Summing on the index k, the second term can be written as,

$$-\sum_k\int d^3r\left[\left(\frac{\partial j_k}{\partial r_k}\right) r_j\, r_i \;+\; j_k\left(\frac{\partial r_j}{\partial r_k}\right) r_i\right] = -\int d^3r\left[(\boldsymbol{\nabla}\cdot\mathbf{j})\, r_j\, r_i \;+\; j_j\, r_i\right] = -\int d^3r\, j_j\, r_i \tag{2.5.15}$$

where we used $(\partial r_j/\partial r_k) = \delta_{jk}$, and $\boldsymbol{\nabla}\cdot\mathbf{j} = 0$. Thus we have that the tensor in ② is antisymmetric,

$$\int d^3r\, j_i\, r_j = -\int d^3r\, j_j\, r_i = \frac{1}{2}\int d^3r\,[j_i\, r_j \;-\; j_j\, r_i] \tag{2.5.16}$$

In the last step we used that if $A = -B$, then $A = \frac{1}{2}A - \frac{1}{2}B$.

Going back to the full term in ② we have

$$\left[\int d^3r'\mathbf{j}\,\mathbf{r}'\right]\cdot\mathbf{r} = \frac{1}{2}\left[\int d^3r'\,(\mathbf{j}\,\mathbf{r}' - \mathbf{r}'\mathbf{j})\right]\cdot\mathbf{r} = \frac{1}{2}\int d^3r'\;[\mathbf{j}(\mathbf{r}'\cdot\mathbf{r}) - \mathbf{r}'(\mathbf{j}\cdot\mathbf{r})] \tag{2.5.17}$$

Recalling the triple product rule, $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$, and identifying $\mathbf{B}\leftrightarrow\mathbf{j}$, $\mathbf{A}\leftrightarrow\mathbf{r}$, and $\mathbf{C}\leftrightarrow\mathbf{r}'$, we have,

$$\int d^3r'\,\mathbf{j}(\mathbf{r}')\,(\mathbf{r}'\cdot\mathbf{r})=\frac{1}{2}\mathbf{r}\times\int d^3r'\,[\mathbf{j}(\mathbf{r}')\times\mathbf{r}']=-\frac{1}{2}\mathbf{r}\times\int d^3r'\,[\mathbf{r}'\times\mathbf{j}(\mathbf{r}')] \tag{2.5.18}$$

We then define the *magnetic dipole moment* as

$$\boxed{\mathbf{m}\equiv\frac{1}{2c}\int d^3r'\,[\mathbf{r}'\times\mathbf{j}(\mathbf{r}')]} \tag{2.5.19}$$

and the second term 2 in the magnetic multipole expansion for the vector potential is,

$$\mathbf{A}^{\text{dipole}}(\mathbf{r})=\frac{-\mathbf{r}\times\mathbf{m}}{r^3}=\frac{\mathbf{m}\times\mathbf{r}}{r^3}=\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \tag{2.5.20}$$

This is the magnetic dipole approximation. What is the magnetic field $\mathbf{B}^{\text{dipole}}$ in this magnetic dipole approximation?

$$\mathbf{B}^{\text{dipole}}=\boldsymbol{\nabla}\times\mathbf{A}^{\text{dipole}}=\boldsymbol{\nabla}\times\left(\mathbf{m}\times\frac{\mathbf{r}}{r^3}\right) \tag{2.5.21}$$

To do the double cross product, it is convenient to recall the Levi-Civita symbol, ε_{ijk}, defined as,

$$\varepsilon_{ijk}=\begin{cases}0 & \text{if any of the two indices } i,j,k \text{ are equal}\\ +1 & \text{if } i,j,k \text{ are an } \textit{even} \text{ permutation of 1,2,3}\\ -1 & \text{if } i,j,k \text{ are an } \textit{odd} \text{ permutation of 1,2,3}\end{cases} \tag{2.5.22}$$

An even permutation is when it takes an even number of pairwise exchanges to map $1,2,3$ onto i,j,k.

In terms of the Levi-Civita symbol, you should confirm that $(\mathbf{A}\times\mathbf{B})_i=\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}A_jB_k$.

We will also use the Einstein summation convention: whenever we have a pair of indices repeated, it is implied that we sum over them, so

$$\varepsilon_{ijk}A_jB_k\equiv\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}A_jB_k \qquad \text{here the indices } j \text{ and } k \text{ are repeated, i.e. they both appear twice} \tag{2.5.23}$$

A *very* useful identity to remember is:

$$\varepsilon_{kij}\varepsilon_{k\ell m}=\delta_{i\ell}\delta_{jm}-\delta_{im}\delta_{j\ell} \qquad \text{(proof left to reader)} \tag{2.5.24}$$

So we will now put this identity to use!

$$\mathbf{B}^{\text{dipole}}=\boldsymbol{\nabla}\times\left(\mathbf{m}\times\frac{\mathbf{r}}{r^3}\right)\quad\Rightarrow \tag{2.5.25}$$

$$(\mathbf{B}^{\text{dipole}})_i=\varepsilon_{ijk}\,\partial_j\,\varepsilon_{k\ell n}\,m_\ell\,\frac{r_n}{r^3} \qquad \text{where } \partial_j\equiv\frac{\partial}{\partial r_j} \tag{2.5.26}$$

$$=\varepsilon_{kij}\varepsilon_{k\ell n}\partial_j\left(\frac{m_\ell r_n}{r^3}\right) \qquad \begin{array}{l}\varepsilon_{ijk}=\varepsilon_{kij} \text{ since an even number of pairwise permutations}\\ \text{takes one from one to the other}\end{array} \tag{2.5.27}$$

$$=(\delta_{i\ell}\delta_{jn}-\delta_{in}\delta_{j\ell})\partial_j\left(\frac{m_\ell r_n}{r^3}\right)=m_i\partial_j\left(\frac{r_j}{r^3}\right)-m_j\partial_j\left(\frac{r_i}{r^3}\right) \qquad \text{summing over } \ell \text{ and } n \tag{2.5.28}$$

$$\tag{2.5.29}$$

Now use $\partial_j(r_j/r^3) = \boldsymbol{\nabla}\cdot(\mathbf{r}/r^3) = \boldsymbol{\nabla}\cdot(\hat{\mathbf{r}}/r^2) = 4\pi\delta(\mathbf{r})$, and

$$\partial_j\left(\frac{r_i}{r^3}\right) = \frac{1}{r^3}\left(\frac{\partial r_i}{\partial r_j}\right) - \frac{3r_i}{r^4}\left(\frac{\partial r}{\partial r_j}\right) \tag{2.5.30}$$

We have $\left(\dfrac{\partial r_i}{\partial r_j}\right) = \delta_{ij}$ and $\left(\dfrac{\partial r}{\partial x}\right) = \dfrac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} = \dfrac{x}{r}$, so $\left(\dfrac{\partial r}{\partial r_j}\right) = \dfrac{r_j}{r}$. Putting it together gives,

$$(\mathbf{B}^{\text{dipole}})_i = 4\pi\delta(\mathbf{r})m_i - m_j\left[\frac{\delta_{ij}}{r^3} - \frac{3r_ir_j}{r^5}\right] \tag{2.5.31}$$

But this is the solution for $\mathbf{r}$ far from the source, and so far from the origin, and so there $\delta(\mathbf{r}) = 0$. Summing over j we get,

$$(\mathbf{B}^{\text{dipole}})_i = \frac{-m_i}{r^3} + \frac{3(\mathbf{m}\cdot\mathbf{r})r_i}{r^5} \tag{2.5.32}$$

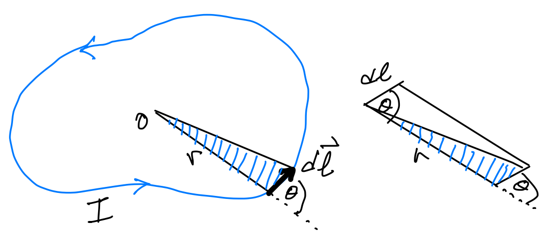
Now using $\mathbf{r} = r\hat{\mathbf{r}}$, we can put it into vector form and get,

$$\boxed{\mathbf{B}^{\text{dipole}}(\mathbf{r}) = \frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3}} \qquad \text{this has exactly the same form as } \mathbf{E}^{\text{dipole}} \text{ with } \mathbf{p}\to\mathbf{m} \tag{2.5.33}$$

The above is the magnetic field from a general localized current distribution, in the magnetic dipole approximation.

You have probably first seen the magnetic dipole moment in connection with a current carrying loop lying flat in a plane. Let's use our expression (2.5.19) to compute $\mathbf{m}$ for this case.

$$\mathbf{m} = \frac{1}{2c}\int d^3r\,[\mathbf{r}\times\mathbf{j}] = \frac{1}{2c}I\oint_C \mathbf{r}\times d\boldsymbol{\ell} \quad \text{where } C \text{ is the loop, and } \mathbf{j}\,d^3r \to I d\boldsymbol{\ell} \text{ for a 1-dimensional loop.} \tag{2.5.34}$$



area of the triangle is $da = \frac{1}{2}r\,d\ell\,\sin\theta = \frac{1}{2}|\mathbf{r}\times d\boldsymbol{\ell}|$. Integrate around the loop to get,

$$\mathbf{m} = \frac{1}{c}I(\text{area})\hat{\mathbf{n}} \tag{2.5.35}$$

where (area) is the area of the loop, and $\hat{\mathbf{n}}$ is the outward pointing unit normal vector to the plane of the loop.

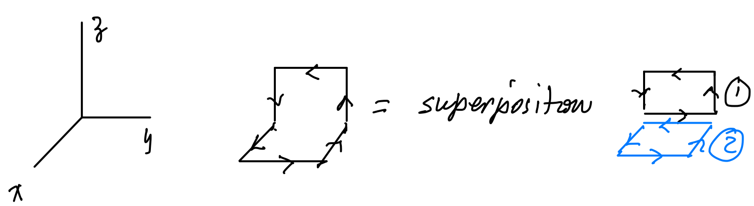
The direction of $\hat{\mathbf{n}}$ must be taken relative to the direction one integrates around the loop according to the right hand rule.

The magnetic dipole moment $\mathbf{m}$ is independent of the location of the coordinate origin. If we transform to new coordinates with $\mathbf{r}' = \mathbf{r}+\mathbf{d}$, then,

$$\mathbf{m}' = \frac{1}{2c}\int d^3r'\,[\mathbf{r}'\times\mathbf{j}] = \frac{1}{2c}\int d^3r\,[(\mathbf{r}+\mathbf{d})\times\mathbf{j} = \frac{1}{2c}\int d^3r\,[\mathbf{r}\times\mathbf{j}] + \frac{1}{2c}\mathbf{d}\times\int d^3r\,\mathbf{j} = \mathbf{m} \tag{2.5.36}$$

where the last step follows because we previously showed that $\displaystyle\int d^3r\,\mathbf{j} = 0$.

We can extend our result for a planar loop to a *piecewise* planar loop (a loop made up of several pieces, each of which is planar).



The loop 1 lies in the yz plane and has magnetic dipole moment $\mathbf{m}_1 = \frac{1}{c} I a_1 \hat{\mathbf{x}}$. The loop 2 lies in the xy plane and has dipole moment $\mathbf{m}_2 = \frac{1}{c} I a_2 \hat{\mathbf{z}}$, where a_1 and a_2 are the areas of the loops. The original piecewise planar loop can be viewed as a superposition of loop 1 and 2 since the currents of these two loops in the segments along the $\hat{\mathbf{y}}$ axis cancel. We therefore have for the magnetic dipole moment of the original loop,

$$\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2 = \frac{1}{c} I [a_1 \hat{\mathbf{x}} + a_2 \hat{\mathbf{z}}] \tag{2.5.37}$$

Boundary Value Problems and the Scalar Magnetic Potential

Because of the *vector* character of the magnetostatic Poisson's equation, $-\nabla^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{j}$, and the fact that $\nabla^2 \mathbf{A}$ only has a convenient representation in Cartesian coordinates, many of the methods we used to solve electrostatic boundary value problems for $-\nabla^2\phi = 4\pi\rho$ do not easily apply to the magnetostatic case.

However, in situations where the current $\mathbf{j}$ is confined to certain current carrying surfaces, we can make the problem much closer to the electrostatic case by using a mathematical trick known as the *scalar* magnetic potential ϕ_M.

In regions where $\mathbf{j} = 0$, i.e. *not* on the current carrying surfaces, the magnetostatic Ampere's Law becomes, $\boldsymbol{\nabla} \times \mathbf{B} = 0$. In such regions we can define a scalar potential ϕ_M such that,

$$\mathbf{B} = -\boldsymbol{\nabla}\phi_M \tag{2.5.38}$$

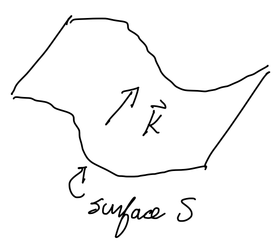
Since we still have $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, then in such current free regions ϕ_M obeys Laplace's equation,

$$\nabla^2 \phi_M = 0 \tag{2.5.39}$$

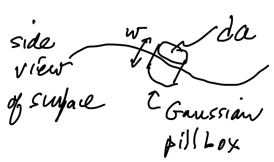
We can then solve for ϕ_M in the current free regions, just like we did in electrostatics, and then match up solutions on opposite sides of the current carrying surfaces by applying appropriate boundary conditions.

Review: Magnetic field at a current carrying surface

In magnetostatics, $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and $\boldsymbol{\nabla} \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j}$



Consider a surface S which has a surface current $\mathbf{K}(\mathbf{r})$ at point $\mathbf{r}$ on the surface.



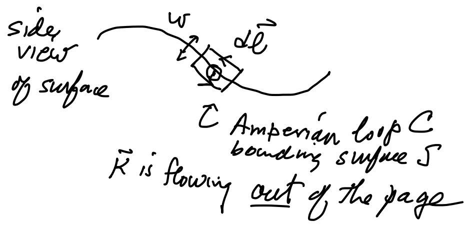
Construct a small Gaussian pillbox that pierces the surface, with top and bottom areas da and width w. Consider the integral over the volume of the pillbox,

$$\int_V d^3r \; \boldsymbol{\nabla} \cdot \mathbf{B} = 0 = \oint_S da \, \hat{\mathbf{n}} \cdot \mathbf{B} \qquad \text{by Gauss' Theorem} \tag{2.5.40}$$

If we let the side width $w \to 0$, then the contribution of the side to the surface integral will vanish, and we are left only with the contributions from the top and bottom sides,

$$\Rightarrow \quad \oint_S da \, \hat{\mathbf{n}} \cdot \mathbf{B} = da \, \hat{\mathbf{n}} \cdot (\mathbf{B}^{\text{above}} - \mathbf{B}^{\text{below}}) = 0 \quad \text{where } \hat{\mathbf{n}} \text{ is the outward normal pointing from below to above} \tag{2.5.41}$$

$$\Rightarrow \quad \hat{\mathbf{n}} \cdot (\mathbf{B}^{\text{above}} - \mathbf{B}^{\textbf{below}}) = 0 \quad \text{the \underline{normal} component of } \mathbf{B} \text{ is \underline{continuous}} \tag{2.5.42}$$



Now construct a small Amperian loop C that pierces the surface, with top and bottom sides of length $d\ell$, and the sides orthogonal to the current carrying surface of length w. The area bounded by C we will call $\mathcal{S}$ (not to be confused with the current carrying surface S).

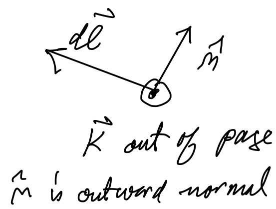
Integrating over the loop area, and applying Stoke's Theorem, we get,

$$\int_{\mathcal{S}} da\, \hat{\mathbf{n}}_s \cdot (\boldsymbol{\nabla} \times \mathbf{B}) = \oint_C \mathbf{B} \cdot \boldsymbol{d\ell} = \frac{4\pi}{c} I_{\text{enclosed}} \tag{2.5.43}$$

where $\hat{\mathbf{n}}_s$ is the normal to the surface $\mathcal{S}$ bounded by the loop C, and I_{enclosed} is the total flux of the current through the surface $\mathcal{S}$, given by $I_{\text{enclosed}} = d\ell\, \hat{\mathbf{n}}_s \cdot \mathbf{K}$.

If we let the side width $w \to 0$, then the contribution to the line integral from those sides will vanish, and we are left with

$$\oint_C \mathbf{B} \cdot \boldsymbol{d\ell} = (\mathbf{B}^{\text{above}} - \mathbf{B}^{\text{below}}) \cdot \boldsymbol{d\ell} = \frac{4\pi}{c} I_{\text{enclosed}} = \frac{4\pi}{c} d\ell\, \hat{\mathbf{n}}_s \cdot \mathbf{K} = \frac{4\pi}{c} (\hat{\mathbf{n}} \times \boldsymbol{d\ell}) \cdot \mathbf{K} = \frac{4\pi}{c} (\mathbf{K} \times \hat{\mathbf{n}}) \cdot \boldsymbol{d\ell} \tag{2.5.44}$$



Here $\hat{\mathbf{n}}$ is the unit normal pointing from below to above the surface, and so $\hat{\mathbf{n}} \times \boldsymbol{d\ell} = d\ell\, \hat{\mathbf{n}}_s$ is the vector of length $d\ell$ pointing perpendicular to the loop C, in the same orientation that the surface current $\mathbf{K}$ is flowing, The last step follows from the vector identity $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}$.

Since $\boldsymbol{d\ell}$ can have *any* direction within the plane of the current carrying surface at position $\mathbf{r}$, we conclude that the tangential component of $\mathbf{B}$ has a discontinuous jump when crossing the surface equal to $\dfrac{4\pi}{c}\mathbf{K} \times \hat{\mathbf{n}}$.

Since $\mathbf{K} \times \hat{\mathbf{n}}$ is tangent to the plane of the current carrying surface, we can combine this with our previous result for the normal component of $\mathbf{B}$ to conclude

$$\boxed{\mathbf{B}^{\text{above}} - \mathbf{B}^{\text{below}} = \frac{4\pi}{c} \mathbf{K} \times \hat{\mathbf{n}}} \tag{2.5.45}$$

This is the magnetic analog of $\mathbf{E}^{\text{above}} - \mathbf{E}^{\text{below}} = 4\pi\sigma\hat{\mathbf{n}}$.

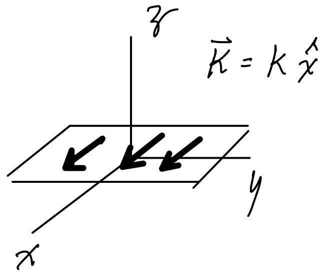
In terms of the scalar magnetic potential ϕ_M this becomes,

$$-\boldsymbol{\nabla}\phi_M^{\text{above}} + \boldsymbol{\nabla}\phi_M^{\text{below}} = \frac{4\pi}{c} \mathbf{K} \times \hat{\mathbf{n}} \tag{2.5.46}$$

Note: ϕ_M is a calculational tool only. It does not have any direct physical significance as does the electrostatic potential ϕ, which is related to the work done to move a charge (in fact, magnetic forces can do no work since $\frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v} = \frac{q}{c}\mathbf{v} \cdot [\mathbf{v} \times \mathbf{B}] = 0$).

Note also: ϕ_M is in general *not* continuous at a current carrying surface, because ϕ_M is not defined on the surface, and so we cannot do similarly to electrostatics to conclude $\phi_M(\mathbf{r}^{\text{above}}) - \phi_M(\mathbf{r}^{\text{below}}) = -\int_{\mathbf{r}^{\text{below}}}^{\mathbf{r}^{\text{above}}} \mathbf{B} \cdot \boldsymbol{d\ell} = 0$. This is because $\mathbf{B} = -\boldsymbol{\nabla}\phi_M$ holds both above and below the surface, but not passing through the surface, because ϕ_M is only defined where the current is zero.

Example: An infinite flat plane at $z=0$ with a surface current $\mathbf{K}=K\hat{\mathbf{x}}$.



$$z>0, \quad \nabla^2\phi_M^+ = 0 \quad \Rightarrow \quad \phi_M^+ = a^+ - b_x^+ x - b_y^+ y - b_z^+ z \tag{2.5.47}$$

$$z<0, \quad \nabla^2\phi_M^- = 0 \quad \Rightarrow \quad \phi_M^- = a^- - b_x^- x - b_y^- y - b_z^- z \tag{2.5.48}$$

$$\Rightarrow \quad z>0, \quad \mathbf{B}^+ = -\boldsymbol{\nabla}\phi_M^+ = b_x^+\hat{\mathbf{x}} + b_y^+\hat{\mathbf{y}} + b_z^+\hat{\mathbf{z}} \tag{2.5.49}$$

$$z<0, \quad \mathbf{B}^- = -\boldsymbol{\nabla}\phi_M^- = b_x^-\hat{\mathbf{x}} + b_y^-\hat{\mathbf{y}} + b_z^-\hat{\mathbf{z}} \tag{2.5.50}$$

At $z=0$, the boundary condition gives,

$$\mathbf{B}^+ - \mathbf{B}^- = (b_x^+ - b_x^-)\hat{\mathbf{x}} + (b_y^+ - b_y^-)\hat{\mathbf{y}} + (b_z^+ - b_z^-)\hat{\mathbf{z}} = \frac{4\pi}{c}\mathbf{K}\times\hat{\mathbf{n}} = \frac{4\pi K}{c}(\hat{\mathbf{x}}\times\hat{\mathbf{z}}) = \frac{-4\pi K}{c}\hat{\mathbf{y}} \tag{2.5.51}$$

$$\Rightarrow \quad b_x^+ = b_x^- \equiv b_{x0}, \qquad b_z^+ = b_z^- \equiv b_{z0}, \qquad b_y^+ - b_y^- = \frac{-4\pi K}{c} \tag{2.5.52}$$

Define $b_{y0} = (b_y^+ + b_y^-)/2$, and $\delta b_y = (b_y^+ - b_y^-)/2$, so that,

$$b_y^+ = b_{y0} + \delta b_y, \qquad b_y^- = b_{y0} - \delta b_y \qquad \Rightarrow \qquad \delta b_y = \frac{-2\pi K}{c} \tag{2.5.53}$$

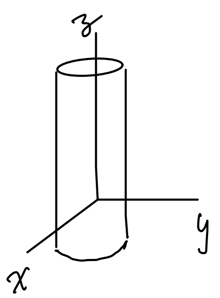
$$\Rightarrow \qquad \mathbf{B}^+ = \mathbf{B}_0 - \frac{2\pi K}{c}\hat{\mathbf{y}}, \qquad \mathbf{B}^- = \mathbf{B}_0 + \frac{2\pi K}{c}\hat{\mathbf{y}}, \qquad \text{where} \quad \mathbf{B}_0 = b_{x0}\hat{\mathbf{x}} + b_{y0}\hat{\mathbf{y}} + b_{z0}\hat{\mathbf{z}} \tag{2.5.54}$$

Here $\mathbf{B}_0$ is a constant magnetic field that fills all the universe. If the surface current $\mathbf{K}$ is the only source of magnetic field, then we should have $\mathbf{B}_0 = 0$, and then

$$\mathbf{B} = \begin{cases} \dfrac{-2\pi K}{c}\hat{\mathbf{y}} & z>0 \\[2ex] \dfrac{2\pi K}{c}\hat{\mathbf{y}} & z<0 \end{cases} \tag{2.5.55}$$

$\mathbf{B}$ is constant in magnitude everywhere, is in direction $\hat{\mathbf{y}}$, but points in opposite directions on either side of the current carrying plane.

Example: An infinite current carrying cylinder of radius R



We consider two cases: (i) $\mathbf{K} = K\hat{\mathbf{z}}$ wire with surface current
(ii) $\mathbf{K} = K\hat{\boldsymbol{\varphi}}$ solenoid

Case (i): current carrying wire

The total current flowing down the wire is $I = 2\pi R K$. We can guess the solution for ϕ_M to $\nabla^2\phi_M = 0$, and then check that it is correct.

$$\left.\begin{array}{ll} r>R, & \phi_M = \dfrac{-2I\varphi}{c} \\[2ex] r<R, & \phi_M = 0 \end{array}\right\} \Rightarrow \begin{array}{ll} r>R, & \mathbf{B} = -\boldsymbol{\nabla}\phi_M = -\dfrac{1}{r}\dfrac{\partial\phi_M}{\partial\varphi}\hat{\boldsymbol{\varphi}} = \dfrac{2I}{cr}\hat{\boldsymbol{\varphi}} \\[2ex] r<R, & \mathbf{B} = -\boldsymbol{\nabla}\phi_M = 0 \end{array} \tag{2.5.56}$$

Check that this satisfies the boundary condition:

$$\mathbf{B}^{\text{above}} - \mathbf{B}^{\text{below}} = \frac{2I}{cR}\hat{\boldsymbol{\varphi}} = \frac{4\pi K R}{cR}\hat{\boldsymbol{\varphi}} = \frac{4\pi K}{c}\hat{\boldsymbol{\varphi}} = \frac{4\pi}{c}\mathbf{K}\times\hat{\mathbf{n}} \qquad \text{since } \mathbf{K}\times\hat{\mathbf{n}} = K\hat{\mathbf{z}}\times\hat{\mathbf{r}} = K\hat{\boldsymbol{\varphi}} \tag{2.5.57}$$

Note, $\phi_M = -(2I/c)\varphi$ is *not* a single valued function! We would not have found this solution using separation of variables in polar coordinates. But ϕ_M does not need to be single valued since it has no direct physical significance, only $\mathbf{B} = -\boldsymbol{\nabla}\phi_M$ is a physical quantity.

Case (ii): infinite solenoid

Again we guess the solution to $\nabla^2\phi_M = 0$ for this problem,

$$\left.\begin{array}{ll} r > R, & \phi_M = -B_1 z \\ r < R, & \phi_M = -B_2 z \end{array}\right\} \quad \Rightarrow \quad \begin{array}{ll} r > R, & \mathbf{B} = -\boldsymbol{\nabla}\phi_M = B_1\hat{\mathbf{z}} \\ r < R, & \mathbf{B} = -\boldsymbol{\nabla}\phi_M = B_2\hat{\mathbf{z}} \end{array} \tag{2.5.58}$$

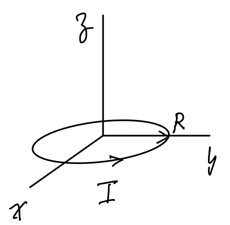
Apply the boundary condition:

$$\mathbf{B}^{\text{above}} - \mathbf{B}^{\text{below}} = (B_1 - B_2)\hat{\mathbf{z}} = \frac{4\pi}{c}\mathbf{K}\times\hat{\mathbf{n}} = \frac{4\pi K}{c}(\hat{\boldsymbol{\varphi}}\times\hat{\mathbf{r}}) = \frac{-4\pi K}{c}\hat{\mathbf{z}} \tag{2.5.59}$$

If the current in the solenoid is the only source of magnetic field, then we expect that the field outside $B_1 = 0$, so then

$$\mathbf{B} = \begin{cases} \dfrac{4\pi K}{c}\hat{\mathbf{z}} & \text{inside, } r < R \\ 0 & \text{outside, } r > R \end{cases} \tag{2.5.60}$$

Example: Circular current loop of radius R in the xy plane



For $r > R$, the current $\mathbf{j} = 0$ so $\boldsymbol{\nabla}\times\mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = -\boldsymbol{\nabla}\phi_M$ and $\nabla^2\phi_M = 0$.

We can therefore write a Legendre polynomial expansion for ϕ_M

$$\phi_M = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}}\,\mathcal{P}_\ell(\cos\theta) \tag{2.5.61}$$

(the $A_\ell r^\ell$ terms vanish as we want $\mathbf{B}\to 0$ as $r\to\infty$)

So,

$$\mathbf{B} = -\boldsymbol{\nabla}\phi_M = -\frac{\partial\phi_M}{\partial r}\,\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\phi_M}{\partial\theta}\,\hat{\boldsymbol{\theta}} = \sum_\ell\left[\frac{(\ell+1)B_\ell}{r^{\ell+2}}\,\mathcal{P}_\ell(\cos\theta)\,\hat{\mathbf{r}} \;-\; \frac{B_\ell}{r^{\ell+2}}\frac{\partial\mathcal{P}_\ell(\cos\theta)}{\partial\theta}\,\hat{\boldsymbol{\theta}}\right] \tag{2.5.62}$$

Write,

$$\frac{\partial\mathcal{P}_\ell}{\partial\theta} = \frac{\partial\mathcal{P}_\ell}{\partial x}\frac{\partial x}{\partial\theta} = -\frac{\partial\mathcal{P}_\ell}{\partial x}\sin\theta = -\mathcal{P}_\ell'\sin\theta \qquad \text{where} \quad x = \cos\theta \tag{2.5.63}$$

So,

$$\mathbf{B} = \sum_\ell\left[\frac{(\ell+1)B_\ell}{r^{\ell+2}}\,\mathcal{P}_\ell(\cos\theta)\,\hat{\mathbf{r}} \;+\; \frac{B_\ell}{r^{\ell+2}}\sin\theta\,\mathcal{P}_\ell'(\cos\theta)\,\hat{\boldsymbol{\theta}}\right] \tag{2.5.64}$$

Along the $\hat{\mathbf{z}}$ axis, $\theta = 0$ and $\mathcal{P}_\ell(1) = 1$, and the above becomes,

$$\mathbf{B}(z\hat{\mathbf{z}}) = \sum_\ell \frac{(\ell+1)B_\ell}{r^{\ell+2}}\,\hat{\mathbf{r}} = \sum_\ell \frac{(\ell+1)B_\ell}{z^{\ell+2}}\,\hat{\mathbf{z}} \tag{2.5.65}$$

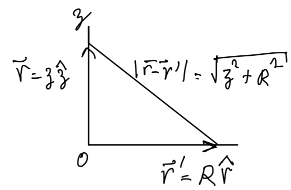
To find the coefficients B_ℓ we can compare to the exact solution on the $\hat{\mathbf{z}}$ axis, obtained as follows:

$$\mathbf{A} = \int\frac{d^3r'}{c}\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \quad \Rightarrow \quad \mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla}\times\mathbf{A} = \int\frac{d^3r'}{c}\,\boldsymbol{\nabla}\times\left[\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] \tag{2.5.66}$$

$$\mathbf{B} = -\int\frac{d^3r'}{c}\,\mathbf{j}(\mathbf{r}')\times\boldsymbol{\nabla}\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right] = \int\frac{d^3r'}{c}\,\mathbf{j}(\mathbf{r}')\times\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \qquad \text{Biot-Savart Law} \tag{2.5.67}$$

For our circular loop, $d^3r'\,\mathbf{j} \to I\boldsymbol{d\ell}'$, and so,

$$\mathbf{B}(\mathbf{r}) = \frac{I}{c}\oint_C \boldsymbol{d\ell}' \times \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{2.5.68}$$



For $\mathbf{r} = z\hat{\mathbf{z}}$ on the $\hat{\mathbf{z}}$ axis, and $\mathbf{r}' = R\hat{\mathbf{r}}$ on the loop, we can write the integral around the loop in polar coordinates,

$$\mathbf{B}(z\hat{\mathbf{z}}) = \frac{I}{c}\int_0^{2\pi} d\varphi\, R\,\hat{\boldsymbol{\varphi}} \times \frac{[z\hat{\mathbf{z}} - R\hat{\mathbf{r}}]}{[z^2+R^2]^{3/2}} \qquad \text{where} \quad \boldsymbol{d\ell}' = d\varphi\, R\hat{\boldsymbol{\varphi}} \tag{2.5.69}$$

Now $\hat{\boldsymbol{\varphi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}} \times \hat{\mathbf{r}} = -\hat{\mathbf{z}}$, so

$$\mathbf{B}(z\hat{\mathbf{z}}) = \frac{I}{c}\int_0^{2\pi} d\varphi\, R\frac{[z\hat{\mathbf{r}} + R\hat{\mathbf{z}}]}{[z^2+R^2]^{3/2}} \tag{2.5.70}$$

The only term in the integrand that depends on the integration variable φ is $\hat{\mathbf{r}}$. We can write $\hat{\mathbf{r}} = \cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}}$, and so when we do the integral, $\int_0^{2\pi} d\varphi \cos\theta = \int_0^{2\pi} d\varphi \sin\theta = 0$, and this term vanishes. We are left with the exact solution for the magnetic field on the $\hat{\mathbf{z}}$ axis,

$$\mathbf{B}(z\hat{\mathbf{z}}) = \frac{2\pi R^2 I}{c[z^2+R^2]^{3/2}}\hat{\mathbf{z}} = \frac{2\pi R^2 I}{cz^3}\,\frac{1}{[1+(R/z)^2]^{3/2}}\,\hat{\mathbf{z}} \tag{2.5.71}$$

To match with the Legendre polynomial expression of Eq. (2.5.65) we do a Taylor series expansion of the last factor in the above for small (R/z),

$$\mathbf{B}(z\hat{\mathbf{z}}) = \frac{2\pi R^2 I}{cz^3}\left[1 - \frac{3}{2}\left(\frac{R}{z}\right)^2 + \dots\right]\hat{\mathbf{z}} = \frac{2\pi R^2 I}{c}\left[\frac{1}{z^3} - \frac{3}{2}\frac{R^2}{z^5} + \dots\right]\hat{\mathbf{z}} \tag{2.5.72}$$

and compare to the Legendre series of Eq. (2.5.65),

$$\mathbf{B}(z\hat{\mathbf{z}}) = \left[\frac{B_0}{r^2} + \frac{2B_1}{z^3} + \frac{3B_2}{z^4} + \frac{4B_3}{z^5} + \dots\right]\hat{\mathbf{z}} \tag{2.5.73}$$

This gives,

$$B_0 = 0, \qquad B_1 = \frac{\pi R^2 I}{c}, \qquad B_2 = 0, \qquad B_3 = -\frac{3\pi R^4 I}{4c} \tag{2.5.74}$$

So to order $\ell = 3$, we get for the magnetic field at any position $\mathbf{r}$,

$$\mathbf{B}(\mathbf{r}) = \frac{\pi R^2 I}{c}\left[\left(\frac{2\mathcal{P}_1(\cos\theta)\hat{\mathbf{r}} + \sin\theta\,\mathcal{P}_1'(\cos\theta)\hat{\boldsymbol{\theta}}}{r^3}\right) - \left(\frac{3R^2\mathcal{P}_3(\cos\theta)\hat{\mathbf{r}} + \frac{3}{4}R^2\sin\theta\,\mathcal{P}_3'(\cos\theta)\hat{\boldsymbol{\theta}}}{r^5}\right)\right] \tag{2.5.75}$$

Use $\mathcal{P}_1(x) = x \;\Rightarrow\; \mathcal{P}_1'(x) = 1$, and $\mathcal{P}_3(x) = \frac{1}{2}(5x^3 - 3x) \;\Rightarrow\; \mathcal{P}_3'(x) = \frac{1}{2}(15x^2 - 3)$ to get,

$$\mathbf{B}(\mathbf{r}) = \frac{\pi R^2 I}{c}\left[\left(\frac{2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}}{r^3}\right) - \left(\frac{\frac{3}{2}R^2(5\cos^3\theta - 2\cos\theta)\,\hat{\mathbf{r}} + \frac{3}{8}R^2\sin\theta(15\cos^2\theta - 3)\,\hat{\boldsymbol{\theta}}}{r^5}\right)\right] \tag{2.5.76}$$

where the prefactor $\pi R^2 I/c = m$ is just the magnitude of the magnetic dipole moment of the loop.

We see that the first term above is just the magnetic dipole approximation for the field from the loop. The second term is the magnetic *octupole* term. In principle we could easily get higher order terms by this method.

You may compare our result above to Jackson Eq. (5.40).