Unit 3-4: Linear Materials

Macroscopic Maxwell Equations

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \qquad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial B}{\partial t} \tag{3.4.1}$$

$$\boldsymbol{\nabla} \cdot \mathbf{D} = 4\pi\rho \qquad \boldsymbol{\nabla} \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \tag{3.4.2}$$

where ρ and $\mathbf{j}$ are the *macroscopic* charge and current densities that do *not* include the bound charges or currents.

$\mathbf{D}$ and $\mathbf{H}$ are related to $\mathbf{E}$ and $\mathbf{B}$ by,

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \qquad \text{with } \mathbf{P} \text{ the polarization density} \tag{3.4.3}$$

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}, \qquad \text{with } \mathbf{M} \text{ the magnetization density} \tag{3.4.4}$$

To close this system of equation we will in general need to know how $\mathbf{P}$ and $\mathbf{M}$ are related to the $\mathbf{E}$ and $\mathbf{B}$ in the material.

In some materials there can be a finite $\mathbf{P}$ or $\mathbf{M}$ even if $\mathbf{E}$ and $\mathbf{B}$ are zero.

Ferromagnet: $\mathbf{M}$ can be non-zero even if $\mathbf{B} = 0$.

Ferroelectric: $\mathbf{P}$ can be non-zero even if $\mathbf{E} = 0$.

But much more common are <u>linear</u> materials in which, for *small* $\mathbf{E}$ and $\mathbf{B}$, one has $\mathbf{P} \propto \mathbf{E}$ and $\mathbf{M} \propto \mathbf{B}$.

Linear Dielectrics

For a linear dielectric, one has

$$\mathbf{P} = \chi_e \mathbf{E} \tag{3.4.5}$$

where χ_e is the *electric susceptibility*. For electrostatics, $\chi_e > 0$. For such a linear dielectric,

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = (1 + 4\pi\chi_e)\mathbf{E} \qquad \Rightarrow \qquad \mathbf{D} = \epsilon\mathbf{E} \quad \text{with } \epsilon = 1 + 4\pi\chi_e \text{ the } \textit{dielectric constant} \tag{3.4.6}$$

Linear Magnetic Materials

For a linear magnetic material, one has,

$$\mathbf{M} = \chi_m \mathbf{H} \tag{3.4.7}$$

where χ_m is the *magnetic susceptibility*. For magnetostatics one may have $\chi_m > 0$ for a paramagnetic material; or one many have $\chi_m < 0$ for a diamagnetic material. For such a linear magnetic material,

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = \mathbf{B} - 4\pi\chi_m\mathbf{H} \quad \Rightarrow \quad \mathbf{B} = (1+4\pi\chi_m)\mathbf{H} \quad \Rightarrow \quad \mathbf{B} = \mu\mathbf{H} \text{ with } \mu = 1 + 4\pi\chi_m \text{ the } \textit{magnetic permeability} \tag{3.4.8}$$

For *statics* χ_e and χ_m, and so ϵ and μ, are *constants* depending on the type of material. When we later consider dynamics, we will see that ϵ becomes a function of *frequency*.

Clausius - Mossotti Equation - electric susceptibility vs atomic polarizability

If an electric field $\mathbf{E}_{\mathrm{loc}}$ is applied to an atom it gets polarized, $\mathbf{p} = \alpha\mathbf{E}_{\mathrm{loc}}$. Here $\mathbf{p}$ is the atomic electric dipole moment, $\mathbf{E}_{\mathrm{loc}}$ is the local electric field acting on the atom, and α is the *atomic polarizability*.

The atomic polarizability α is something you might think of trying to compute in some microscopic theory (classical or quantum) of how an individual atom behaves when it is placed in an electric field. In contrast, the electric susceptibility χ_e is giving the macroscopic behavior of the material, consisting of many such atoms; it is the parameter one would more easily measure in an experiment. We would like to know how the macroscopic χ_e is related to the microscopic α.

In the simplest picture, one might assume that the local field acting on the atom is just the average macroscopic electric field in the material, $\mathbf{E}_{\text{loc}} = \mathbf{E}$. Then, if n is the density of polarizable atoms in the material, one would have,

$$\mathbf{P} = n\mathbf{p} = n\alpha\mathbf{E}_{\text{loc}} = n\alpha\mathbf{E} = \chi_e\mathbf{E} \qquad \text{so that} \qquad \chi_e = n\alpha \tag{3.4.9}$$

This turns out to be not too bad for a dilute system, such as a dilute dielectric gas. But a more careful consideration shows that $\mathbf{E}_{\text{loc}} \neq \mathbf{E}$. Including this correction leads to an improved relation between χ_e and α, known as the Clausisu-Mossotti equation.

Consider a particular atom, for which we want to see how much it polarizes. The reason why $\mathbf{E}_{\text{loc}} \neq \mathbf{E}$ is because the average electric field in the material $\mathbf{E}$ includes a contribution from the polarized atom under consideration. That contribution should not be included when determining the electric field that polarizes the atom, since the charges in that atom cannot polarize themselves! So the $\mathbf{E}_{\text{loc}}$ that the atom sees, and that causes the atom to polarize, should exclude the contribution to the average $\mathbf{E}$ that comes from the atom's own electric field. We can write,

$$\mathbf{E} = \mathbf{E}_{\text{loc}} + \mathbf{E}_{\text{atom}} \tag{3.4.10}$$

where $\mathbf{E}_{\text{atom}}$ is the electric field produced by the polarized atom under consideration.

Cut out sphere of volume $1/n$
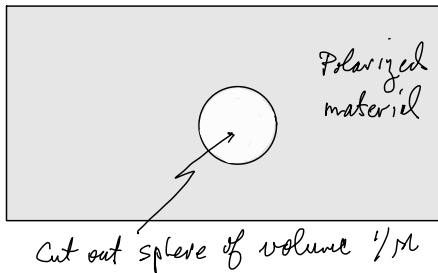
To estimate the average field produced by the polarized atom, $\mathbf{E}_{\text{atom}}$, we can represent the atom by a uniformly polarized sphere of volume $1/n$, since $1/n$ is the volume per particle when the particle density is n. Now we know that the electric field inside a sphere of uniform polarization $\mathbf{P}$ is just $-(4\pi/3)\mathbf{P}$, so we take,

$$\mathbf{E}_{\text{atom}} = -\frac{4\pi}{3}\mathbf{P} \tag{3.4.11}$$

Now,

$$\mathbf{E}_{\text{loc}} = \mathbf{E} - \mathbf{E}_{\text{atom}} = \mathbf{E} + \frac{4\pi}{3}\mathbf{P} \tag{3.4.12}$$

Since $\mathbf{p} = \alpha\mathbf{E}_{\text{loc}}$ we then have,

$$\mathbf{P} = n\mathbf{p} = n\alpha\mathbf{E}_{\text{loc}} = n\alpha\left(\mathbf{E} + \frac{4\pi}{3}\mathbf{P}\right) = n\alpha\mathbf{E} + \frac{4\pi}{3}n\alpha\mathbf{P} \tag{3.4.13}$$

So, solving for $\mathbf{P}$ we get,

$$\mathbf{P} = \frac{n\alpha}{1 - \dfrac{4\pi n\alpha}{3}}\mathbf{E} = \chi_e\,\mathbf{E} \qquad \Rightarrow \qquad \boxed{\chi_e = \frac{n\alpha}{1 - \dfrac{4\pi n\alpha}{3}}} \tag{3.4.14}$$

Note, as n gets very small, this becomes $\chi_e \approx n\alpha$, and so the simple picture we started with will be ok for a very dilute system.

We can now rewrite the above result in terms of the dielectric constant ϵ rather than χ_e. Rewriting the above as an equation for α in terms of χ_e, we have,

$$\alpha = \frac{\chi_e}{n\left(1 + \dfrac{4\pi\chi_e}{3}\right)} \tag{3.4.15}$$

Then using $\epsilon = 1 + 4\pi\chi_e$, so that $\chi_e = (\epsilon - 1)/4\pi$, we get

$$\alpha = \frac{\epsilon - 1}{4\pi n} \frac{1}{\left(1 + \dfrac{\epsilon - 1}{3}\right)} \qquad \Rightarrow \qquad \boxed{\alpha = \frac{3}{4\pi n}\left(\frac{\epsilon - 1}{\epsilon + 2}\right)} \tag{3.4.16}$$

which is called the Clausius-Mossotti equation; in optics this result is also known as the Lorentz-Lorenz equation.

To get a more physical feeling for the Clausius-Mossotti equation, we can take a simple classical picture for the polarization of an atom. Suppose the atom is a point positive ion $+q$ at the origin, surrounded by a spherical electron cloud of radius a and uniform charge density $\rho = -q/(4\pi a^3/3)$. You should be able to show that the electric field inside the uniform electron cloud is just

$$\mathbf{E}(\mathbf{r}) = \frac{4\pi\rho}{3} r\,\hat{\mathbf{r}} \qquad \text{increases linearly in the radial distance } r \tag{3.4.17}$$

When an external field $\mathbf{E}_0$ is applied to the atom, the electron will get displaced a distance $\mathbf{d}$ such that the force from $\mathbf{E}_0$ and the resorting force from the ion exactly cancel,

$$-q\mathbf{E}_0 - q\mathbf{E}(\mathbf{d}) = 0 \qquad \Rightarrow \qquad \mathbf{E}_0 = -\frac{4\pi\rho}{3}\mathbf{d} = \frac{q}{a^3}\mathbf{d} = \frac{1}{a^3}(q\mathbf{d}) = \frac{1}{a^3}\mathbf{p} \tag{3.4.18}$$

So

$$\mathbf{p} = a^3\,\mathbf{E}_0 \qquad \Rightarrow \qquad \boxed{\alpha = a^3} \tag{3.4.19}$$

From Eq. (3.4.14) we then get

$$\chi_e = \frac{na^3}{1 - \dfrac{4\pi na^3}{3}} \tag{3.4.20}$$

If we let $f = (4\pi a^3/3)n$, and recall that $1/n$ is just the volume per atom, then f is the fraction of the material's volume that is occupied by the atoms of radius a. We can then write,

$$\chi_e = \frac{1}{4\pi}\frac{3f}{1 - f} \tag{3.4.21}$$

As f increases, and the material contains less empty space, the correction due to Clausius and Mossatti gets larger.

Bound Charge

For linear dielectrics, the bound charge ρ_b is always proportional to the free charge ρ (unless otherwise stated, we will assume that our material is made up of neutral particles, and so the macroscopic charge and current densities are just the same as the free charge and current densities).

To see this,

$$\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\boldsymbol{\nabla}\cdot(\chi_e\mathbf{E}) = -\boldsymbol{\nabla}\cdot\left(\frac{\chi_e}{\epsilon}\mathbf{D}\right) \tag{3.4.22}$$

If χ_e (and hence $\epsilon = 1 + 4\pi\chi_e$) is spatially constant, then we can factor it outside the divergence to get,

$$\rho_b = -\frac{\chi_e}{\epsilon}\boldsymbol{\nabla}\cdot\mathbf{D} = -\frac{\chi_e}{\epsilon}4\pi\,\rho \tag{3.4.23}$$

so,

$$\boxed{\rho_b = \frac{-4\pi\chi_e}{1 + 4\pi\chi_e}\rho} \qquad \text{thus where the free charge } \rho = 0, \text{ then also } \rho_b = 0. \tag{3.4.24}$$

The total charge density is,

$$\rho_{\text{tot}} = \rho + \rho_b = \rho \left[1 - \frac{4\pi\chi_e}{1 + 4\pi\chi_e} \right] = \frac{\rho}{1 + 4\pi\chi_e} = \frac{\rho}{\epsilon} \tag{3.4.25}$$

so

$$\boxed{\rho_{\text{tot}} = \frac{\rho}{\epsilon}} \tag{3.4.26}$$

One says that the bound charge "screens" the free charge, so that the total charge is reduced compared to the free charge (since in statics $\chi_e > 0$, one always has $\epsilon > 1$).

Behavior at Interfaces

To recap, for electrostatics in dielectric materials, the Macroscopic Maxwell Equations become,

$$\boldsymbol{\nabla} \cdot \mathbf{D} = 4\pi\rho, \qquad \boldsymbol{\nabla} \times \mathbf{E} = 0 \qquad \text{for } \textit{electrostatics} \tag{3.4.27}$$

For linear materials,

$$\mathbf{D} = \epsilon\mathbf{E} \qquad \Rightarrow \qquad \boldsymbol{\nabla} \cdot (\epsilon\mathbf{E}) = 4\pi\rho \tag{3.4.28}$$

If ϵ were constant in space, then we can factor it out of the divergence to get,

$$\epsilon\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho \qquad \Rightarrow \qquad \boxed{\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{4\pi\rho}{\epsilon} = 4\pi\rho_{\text{tot}} \quad \text{and} \quad \boldsymbol{\nabla} \times \mathbf{E} = 0} \tag{3.4.29}$$

Thus, electrostatics in the dielectric would look just like ordinary electrostatics in a vacuum, except one takes $\rho \to \rho_{\text{tot}}$ as the charge source.

Alternatively, one could write $\mathbf{E} = \mathbf{D}/\epsilon$, and then for a spatially constant ϵ,

$$\boldsymbol{\nabla} \times \mathbf{E} = 0 \qquad \Rightarrow \qquad \boldsymbol{\nabla} \times (\mathbf{D}/\epsilon) = \frac{1}{\epsilon}\boldsymbol{\nabla} \times \mathbf{D} = 0 \qquad \Rightarrow \qquad \boxed{\boldsymbol{\nabla} \times \mathbf{D} = 0 \quad \text{and} \quad \boldsymbol{\nabla} \cdot \mathbf{D} = 4\pi\rho} \tag{3.4.30}$$

which again looks just like ordinary electrostatics in a vacuum, except with the free charge ρ as the source and $\mathbf{E} \to \mathbf{D}$.

If this were all there was, then electrostatics in dielectrics would be essentially no different from electrostatics in a vacuum, except for the above substitutions (i.e., $\rho \to \rho_{\text{tot}}$, or $\mathbf{E} \to \mathbf{D}$). However, complications arise at the interface between a dielectric and the vacuum, or at the interface between two different dielectrics, because at such an interface ϵ changes. At an interface, therefore, $\boldsymbol{\nabla} \times \mathbf{E} = \boldsymbol{\nabla} \times (D/\epsilon) \neq (\boldsymbol{\nabla} \times \mathbf{D})/\epsilon$, and so one does not have $\boldsymbol{\nabla} \times \mathbf{D} = 0$.

What we do in such a situation is to solve for $\mathbf{E}$ or $\mathbf{D}$ inside each dielectric (or vacuum) separately, and then use the boundary conditions,

$$\hat{\mathbf{n}} \cdot (\mathbf{D}^{\text{above}} - \mathbf{D}^{\text{below}}) = 4\pi\sigma, \qquad \hat{\mathbf{t}} \cdot (\mathbf{E}^{\text{above}} - \mathbf{E}^{\text{below}}) = 0 \tag{3.4.31}$$

to match the solutions across the interfaces. Here $\hat{\mathbf{n}}$ is the unit normal to the interface, and $\hat{\mathbf{t}}$ is any unit tangent vector in the plane of the interface.

A similar story holds for linear magnetic materials.

Examples

1) <u>Parallel plate capacitor filled with a dielectric</u>

Two infinite, parallel, flat planes are separated by a distance d, have surface free charge densities $+\sigma$ and $-\sigma$, and are filled with a dielectric material in between. What is the electric field between the plates?
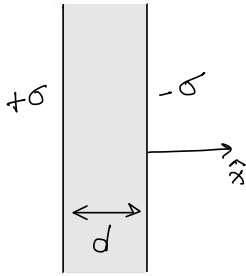
We know that $\mathbf{E} = \mathbf{D} = 0$ outside the plates where there is no dielectric. Between the plates $\boldsymbol{\nabla} \cdot \mathbf{D} = 0$ since there is no free charge there, $\rho = 0$. By symmetry we expect that $\mathbf{D}$ depends only on the coordinate x and points in the $\hat{\mathbf{x}}$ direction, $\mathbf{D}(\mathbf{r}) = D(x)\,\hat{\mathbf{x}}$.

$$\boldsymbol{\nabla} \cdot \mathbf{D} = 0 \quad \Rightarrow \quad \frac{\partial D}{\partial x} = 0 \quad \Rightarrow \quad D = \text{ constant} \tag{3.4.32}$$

$\mathbf{D}$ must satisfy the boundary condition $\hat{\mathbf{n}} \cdot (\mathbf{D}^{\text{above}} - \mathbf{D}^{\text{below}}) = 4\pi\sigma$ at each plane.

At the left side plane with $+\sigma$, with $\hat{\mathbf{n}} = \hat{\mathbf{x}}$ so that the "above" side is to the right, the above boundary condition becomes,

$$D - 0 = 4\pi\sigma \quad \Rightarrow \quad D = 4\pi\sigma \tag{3.4.33}$$

At the right side plane with $-\sigma$, again taking $\hat{\mathbf{n}} = \hat{\mathbf{x}}$, the boundary condition is,

$$0 - D = 4\pi(-\sigma) \quad \Rightarrow \quad D = 4\pi\sigma \tag{3.4.34}$$

so the boundary conditions at both planes give consistent results for D, giving $\mathbf{D} = 4\pi\sigma\hat{\mathbf{x}}$. The electric field between the plates is therefore,

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{4\pi\sigma}{\epsilon}\hat{\mathbf{x}} \tag{3.4.35}$$

Thus the electric field between the plates is reduced by a factor $1/\epsilon$ compared to what it would be if there was a vacuum between the plates.

For more complicated boundary value problems (such as you will see in the homework!) you can apply the techniques we learned for earlier electrostatic problems, such as separation of variables, only now you have to use the appropriate dielectric boundary conditions involving the normal component of $\mathbf{D}$ rather than $\mathbf{E}$.

Similarly one can solve boundary value problems involving magnetic materials by using the same methods we learned earlier, such as the scalar magnetic potential, only now you have to use the appropriate magnetic material boundary conditions involving the tangential component of $\mathbf{H}$ rather than $\mathbf{B}$.

2) Point charge at the center of a dielectric sphere

A point charge q lies at the center of a dielectric sphere of radius R. What is the electric field inside and outside the sphere? What is the bound charge σ_b on the surface of the sphere?
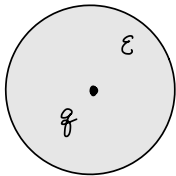
We know that $\boldsymbol{\nabla} \cdot \mathbf{D} = 4\pi\rho$, with the free charge $\rho = q\delta(\mathbf{r})$. In integral form this becomes,

$$\oint_S da\,\hat{\mathbf{n}} \cdot \mathbf{D} = 4\pi Q_{\text{encl}} \qquad \text{where } Q_{\text{encl}} \text{ is the enclosed } \textit{free} \text{ charge} \tag{3.4.36}$$

From rotational symmetry we expect that $\mathbf{D}$ depends only on the radial coordinate and points in the $\hat{\mathbf{r}}$ direction, $\mathbf{D}(\mathbf{r}) = D(r)\,\hat{\mathbf{r}}$.

Applying Eq. (3.4.36) by integrating over a concentric sphere of radius r, we have,

$$\oint_S da\,\hat{\mathbf{n}} \cdot \mathbf{D} = 4\pi r^2 D(r) = 4\pi q \quad \Rightarrow \quad \mathbf{D}(\mathbf{r}) = \frac{q}{r^2}\,\hat{\mathbf{r}} \qquad \text{both inside and outside} \tag{3.4.37}$$

To get the electric field, we know that $\mathbf{E} = \mathbf{D}$ outside the sphere, where there is no dielectric, and $\mathbf{E} = \mathbf{D}/\epsilon$ inside the sphere. So,

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{q}{\epsilon r^2}\,\hat{\mathbf{r}} & r < R \\[2mm] \dfrac{q}{r^2}\,\hat{\mathbf{r}} & r > R \end{cases} \tag{3.4.38}$$

Clearly the tangential component of $\mathbf{E}$ is continuous at the surface (it is zero!), and the normal component of $\mathbf{D}$ is continuous at the surface (since there is no free charge on the surface).

One way to determine the bound surface charge σ_b is to look at the jump in the normal component of $\mathbf{E}$, which is discontinuous by an amount determined by the total surface charge. Since there is no free charge on the surface, the total charge is just the bound charge,

$$\hat{\mathbf{n}} \cdot (\mathbf{E}^{\text{above}} - \mathbf{E}^{\text{below}}) = 4\pi\sigma_{\text{tot}} = 4\pi\sigma_b \tag{3.4.39}$$

From Eq. (3.4.38) we have,

$$\hat{\mathbf{n}} \cdot (\mathbf{E}^{\text{above}} - \mathbf{E}^{\text{below}}) = \hat{\mathbf{r}} \cdot \left(\frac{q}{R^2}\hat{\mathbf{r}} - \frac{q}{\epsilon R^2}\hat{\mathbf{r}}\right) = \frac{q}{R^2}\left(1 - \frac{1}{\epsilon}\right) = \frac{q}{R^2}\left(\frac{\epsilon - 1}{\epsilon}\right) = \frac{q}{R^2}\left(\frac{4\pi\chi_e}{1 + 4\pi\chi_e}\right) = 4\pi\sigma_b \tag{3.4.40}$$

so,

$$\sigma_b = \frac{q}{R^2}\left(\frac{\chi_e}{1 + 4\pi\chi_e}\right) = \frac{q}{R^2}\frac{\chi_e}{\epsilon} \tag{3.4.41}$$

Another way to determine σ_b is to compute the polarization $\mathbf{P}$ and then use $\sigma_b = \hat{\mathbf{n}} \cdot \mathbf{P}$. We have,

$$\mathbf{P} = \chi_e \mathbf{E} = \frac{\chi_e}{\epsilon}\frac{q}{r^2}\hat{\mathbf{r}} \qquad \text{inside the dielectric sphere} \tag{3.4.42}$$

The bound volume charge density is,

$$\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P} = -\frac{\chi_e}{\epsilon}\boldsymbol{\nabla} \cdot \left(\frac{q}{r^2}\right) = -\frac{\chi_e}{\epsilon}4\pi q\delta(\mathbf{r}) = -\frac{4\pi\chi_e}{\epsilon}\rho \tag{3.4.43}$$

with $\rho = q\delta(\mathbf{r})$ the free charge density. Thus the bound volume charge density is just a point charge q_b at the origin, related to the free charge q at the origin by $q_b = -(4\pi\chi_e/\epsilon)q$, as we earlier saw in Eq. (3.4.24). The total charge at the origin is,

$$q_{\text{tot}} = q + q_b = q\left(1 - \frac{4\pi\chi_e}{\epsilon}\right) = q\left(\frac{\epsilon - 4\pi\chi_e}{\epsilon}\right) = \frac{q}{\epsilon} \qquad \text{since } \epsilon = 1 + 4\pi\chi_e \tag{3.4.44}$$

The bound surface charge is,

$$\sigma_b = \hat{\mathbf{n}} \cdot \mathbf{P} = \hat{\mathbf{r}} \cdot \left(\frac{\chi_e}{\epsilon}\frac{q}{R^2}\hat{\mathbf{r}}\right) = \frac{\chi_e}{\epsilon}\frac{q}{R^2} \tag{3.4.45}$$

just as we found earlier from the jump in the normal component of $\mathbf{E}$.

Note, inside the dielectric sphere the $\mathbf{E}$ is just that of the screened point charge q/ϵ at the origin. Outside the dielectric sphere the $\mathbf{E}$ is just that of the free point charge q. Outside the dielectric there is no evidence that the dielectric sphere exists at all!