Unit 3-6: Bar Magnets in Magnetostatics

In this section we consider ferromagnetic bar magnets, where the material has a fixed magnetization density $\mathbf{M}$, even when $\mathbf{B} = 0$. The discussion here will illustrate some of the differences between $\mathbf{B}$ and $\mathbf{H}$.

For a bar magnet, one has $\mathbf{j} = 0$, but $\mathbf{M}$ is fixed and given (this is not a linear material, but rather a ferromagnet!). So for magnetostatics the Macroscopic Maxwell Equations are

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \qquad \text{and} \qquad \boldsymbol{\nabla}\times\mathbf{H} = \frac{4\pi}{c}\mathbf{j} = 0 \qquad \text{since } \mathbf{j} = 0 \tag{3.6.1}$$

Since $\boldsymbol{\nabla}\times\mathbf{H} = 0$ we can write $\mathbf{H} = -\boldsymbol{\nabla}\phi_M$, with ϕ_M the scalar magnetic potential.

Since $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$,

$$\boldsymbol{\nabla}\cdot\mathbf{B} = \boldsymbol{\nabla}\cdot(\mathbf{H} + 4\pi\mathbf{M}) = 0 \qquad \Rightarrow \qquad \boldsymbol{\nabla}\cdot\mathbf{H} = -\nabla^2\phi_M = -4\pi\boldsymbol{\nabla}\cdot\mathbf{M} \tag{3.6.2}$$

So

$$\nabla^2\phi_M = 4\pi\boldsymbol{\nabla}\cdot\mathbf{M} \tag{3.6.3}$$

This is Poisson's equation! Looks just like electrostatics with"magnetic charge density" $\rho_M = -\boldsymbol{\nabla}\cdot\mathbf{M}$.

ρ_M is the source for $\mathbf{H}$.

Note, $\rho_M = -\boldsymbol{\nabla}\cdot\mathbf{M}$, is analogous to our expression for the bound charge density in terms of the polarization density, $\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P}$. Thus, continuing this analogy, one can argue that on the surface of the bar magnet, there is also a "magnetic surface charge density" $\sigma_M = \hat{\mathbf{n}}\cdot\mathbf{M}$.

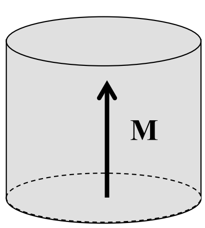
Hence we can solve the Poisson's equation (3.6.3) by integrating over the source "charge."

$$\mathbf{H}(\mathbf{r}) = \int_V d^3r'\,\rho_M(\mathbf{r}')\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} + \oint_S da'\,\sigma_M(\mathbf{r}')\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{3.6.4}$$

where V is the volume of the bar magnet, and S is its surface.

The field lines for $\mathbf{H}$ can start and end at sources and sinks given by ρ_M and σ_M. In contrast, the field lines for $\mathbf{B}$ must still be continuous with no sources or sinks, because we still have $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$.

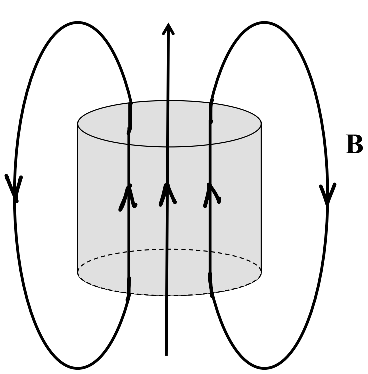
Consider a cylindrical bar magnet of radius R and height L, with fixed $\mathbf{M} = M\hat{\mathbf{z}}$ directed along the cylinder axis.



The magnetization density leads to bound currents flowing in the magnet,

$$\mathbf{j}_b = c\boldsymbol{\nabla}\times\mathbf{M} = 0 \quad \text{but} \quad \mathbf{K}_b = c\mathbf{M}\times\hat{\mathbf{n}} = \begin{cases} cM\hat{\boldsymbol{\varphi}} & \text{on the side} \\ 0 & \text{on the top and bottom} \end{cases} \tag{3.6.5}$$

So $\mathbf{K}_b$ looks just like a solenoidal current flowing around the cylinder side, and the field lines of $\mathbf{B}$ will look as in the sketch below.

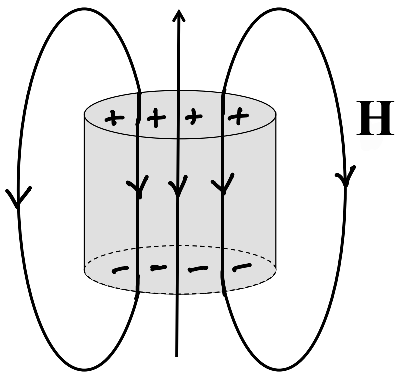


As we see, the field lines of $\mathbf{B}$ are continuous, and either start at infinity and go off to infinity, or close back on themselves.

Now let's look at the same situation from the perspective of $\mathbf{H}$. The field $\mathbf{H}$ is determined from the "magnetic charges" ρ_M and σ_M which are,

$$\rho_M = -\boldsymbol{\nabla}\cdot\mathbf{M} = 0 \qquad \text{and} \qquad \sigma_M = \hat{\mathbf{n}}\cdot\mathbf{M} = \begin{cases} M & \text{on the top} \\ -M & \text{on the bottom} \\ 0 & \text{on the side} \end{cases} \tag{3.6.6}$$

So now the field lines of $\mathbf{H}$ look just like those of a parallel plate capacitor!



The top surface with $\sigma_M = +M$ acts as source for $\mathbf{H}$ field lines, while the bottom surface with $\sigma_M = -M$ acts as a sink for $\mathbf{H}$ field lines.

Outside the bar magnet $\mathbf{H} = \mathbf{B}$, but inside the bar magnet $\mathbf{H}$ and $\mathbf{B}$ are *oppositely* directed!