Unit 5-2: Transparent Propagation, Resonant Absorption, Total Reflection

In this section we will take our model for $\epsilon(\omega)$ of the previous Notes 5-1, and see what are the consequences for EM wave propagation in a dielectric. We use

$$\epsilon(\omega) = 1 + 4\pi\chi_e(\omega) \approx 1 + 4\pi n\alpha(\omega) \tag{5.2.1}$$

where n is the density of polarizable atoms (or molecules) in the material (and *not* the index of refraction!), and $\alpha(\omega)$ is from our simple model of a polarizable atom of Eq. (5.1.17). We then have,

$$\epsilon(\omega) = 1 + \frac{4\pi ne^2}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} \tag{5.2.2}$$

$$\Rightarrow \quad \text{Re}[\epsilon] = \epsilon_1 = 1 + \frac{4\pi ne^2}{m}\frac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2} \tag{5.2.3}$$

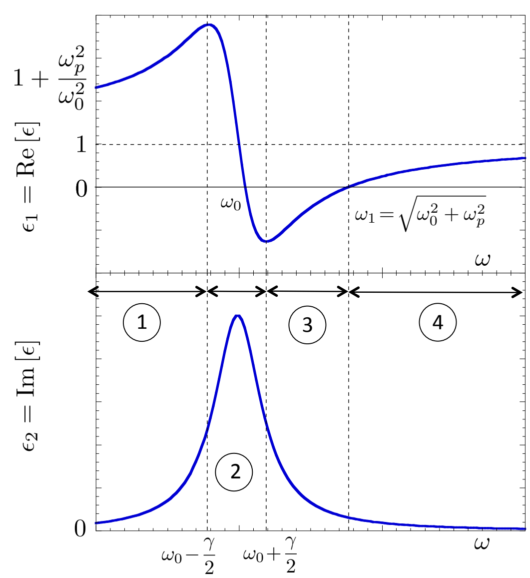
$$\text{Im}[\epsilon] = \epsilon_2 = \frac{4\pi ne^2}{m}\frac{\omega\gamma}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2} \tag{5.2.4}$$

The factor that appears in these terms, $4\pi ne^2/m$, has the units of frequency squared. We define

$$\omega_p \equiv \sqrt{\frac{4\pi ne^2}{m}} \qquad \text{the } \textit{plasma frequency} \tag{5.2.5}$$

We will discuss the various physical significances of the plasma frequency in the following.

We plot $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$ below. These curves have the typical shape of a resonance.



For a sharp resonance, $\gamma \ll \omega_0$, some benchmark features are:

$$\begin{array}{ll}
\epsilon_1 = 1 & \text{when } \omega = \omega_0 \\
\epsilon_1 & \text{has maxima and minima at } \omega \approx \omega_0 \pm \dfrac{\gamma}{2} \\
\epsilon_1 = 1 + \dfrac{\omega_p^2}{\omega_0^2} & \text{when } \omega \to 0 \\
\epsilon_1 \to 1 & \text{when } \omega \to \infty \\
\epsilon_1 = 0 & \text{when } \omega = \omega_1 = \sqrt{\omega_0^2+\omega_p^2} \text{ and } \omega = \omega_0 + \dfrac{\omega_0\gamma^2}{\omega_p^2} \\
\epsilon_2 & \text{has its maximum when } \omega = \bar{\omega} \equiv \sqrt{\omega_0^2 - \gamma^2/4} \approx \omega_0 \\
\epsilon_2 \approx \dfrac{\omega_p^2}{\omega_0\gamma} & \text{at its maximum} \\
\epsilon_2 & \text{has a peak of width } \gamma
\end{array} \tag{5.2.6}$$

(at the end of these notes is the algebra that determines the above)

Now since $\epsilon = \epsilon_1 + i\epsilon_2$ is complex valued, then so is the wavenumber $k = k_1 + ik_2$. We have from the dispersion relation $k = (\omega/c)\sqrt{\mu\epsilon}$,

$$k = k_1 + ik_2 = \frac{\omega}{c}\sqrt{\mu}\sqrt{\epsilon_1 + i\epsilon_2} \quad \Rightarrow \quad k^2 = k_1^2 - k_2^2 + 2ik_1k_2 = \frac{\omega^2}{c^2}\mu(\epsilon_1 + i\epsilon_2) \tag{5.2.7}$$

Note, in the above we squared k, we did not take its absolute value squared!

We can now equate the real parts and the imaginary parts on both sides of the above equation. This gives two equations for the two unknowns, k_1 and k_2. We can then solve for k_1 and k_2 to get,

$$k_1 = \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\sqrt{\epsilon_1^2+\epsilon_2^2}+\frac{1}{2}\epsilon_1\right]^{1/2} \qquad \text{and} \qquad k_2 = \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\sqrt{\epsilon_1^2+\epsilon_2^2}-\frac{1}{2}\epsilon_1\right]^{1/2} \tag{5.2.8}$$

Regions of Different Behavior

Using Eq. (5.2.8) we can classify wave propagation in a dielectric into four different regions of behavior, as indicated in the plot above.

Regions (1) and (4): $\boxed{\epsilon_1 > 0 \text{ and } \epsilon_1 \gg \epsilon_2 \quad \Rightarrow \quad \text{Transparent Propagation}}$

In these regions we have $\epsilon_1 > 0$ and $\epsilon_1 \gg \epsilon_2$. Because of the latter condition, we can expand the square roots in Eq. (5.2.8) in small ϵ_2/ϵ_1. Using $\sqrt{1+\delta} \approx 1+\delta/2$ we get,

$$k_1 \approx \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\epsilon_1\left[1+\frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1}\right)^2\right]+\frac{1}{2}\epsilon_1\right]^{1/2} = \frac{\omega}{c}\sqrt{\mu}\left[\epsilon_1+\frac{1}{4}\frac{\epsilon_2^2}{\epsilon_1}\right]^{1/2} = \frac{\omega}{c}\sqrt{\mu\epsilon_1}\left[1+\frac{1}{4}\left(\frac{\epsilon_2}{\epsilon_1}\right)^2\right]^{1/2} \tag{5.2.9}$$

$$\approx \frac{\omega}{c}\sqrt{\mu\epsilon_1}\left[1+\frac{1}{8}\left(\frac{\epsilon_2}{\epsilon_1}\right)^2\right] = \frac{\omega}{c}\sqrt{\mu\epsilon_1} + \text{ small correction} \tag{5.2.10}$$

and similarly

$$k_1 \approx \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\epsilon_1\left[1+\frac{1}{2}\left(\frac{\epsilon_2}{\epsilon_1}\right)^2\right]-\frac{1}{2}\epsilon_1\right]^{1/2} = \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{4}\frac{\epsilon_2^2}{\epsilon_1}\right]^{1/2} = \frac{\omega}{c}\sqrt{\mu\epsilon_1}\left(\frac{\epsilon_2}{2\epsilon_1}\right) \tag{5.2.11}$$

$$\approx k_1\left(\frac{\epsilon_2}{2\epsilon_1}\right) \ll k_1 \qquad \text{since } \epsilon_2 \ll \epsilon_1 \tag{5.2.12}$$

So in regions (1) and (4) we have $\boxed{k_2 \ll k_1}$. Since the wave goes as $\mathrm{e}^{-k_2 z}\mathrm{e}^{(ik_1 z-\omega t)}$, within one wavelength $\lambda = 2\pi/k_1$ of propagation, the amplitude of the wave has decayed by a factor $\mathrm{e}^{-2\pi k_2/k_1} \approx (1-2\pi k_2/k_1)$, and so there is very little attenuation – the amplitude of wave decays very little for each wavelength of propagation into the material. We say that the medium is transparent (we can see through it!).

Note, in these regions the phase velocity $v_p = \dfrac{\omega}{k_1} = \dfrac{c}{n} = \dfrac{c}{\sqrt{\mu\epsilon_1}}$, where $n = \sqrt{\mu\epsilon_1}$ is the index of refraction.

Assuming $\mu \approx 1$ for a dielectric with only a weak magnetic response, then in region (1) where $\epsilon_1 > 1 \Rightarrow v_p < c$, but in region (2) where $\epsilon_1 < 1 \Rightarrow v_p > c$. But we will always have that the group velocity obeys $v_g < c$.

Also note that in regions (1) and (4) we have $d\epsilon_1/d\omega > 0 \Rightarrow dn/d\omega > 0$, and so these are regions of *normal dispersion*. As one crosses from region (1) into region (2), but does not go far so that we still have $\epsilon_2 \lesssim \epsilon_1$, we have $d\epsilon_1/d\omega < 0 \Rightarrow dn/d\omega < 0$, and so this is a region of *anomalous dispersion*.

Region (2): $\boxed{\omega \approx \omega_0 \quad \Rightarrow \quad \text{Resonant Absorption}}$

In region (2) we are near the peak of ϵ_2, and so $\epsilon_2 \approx \dfrac{\omega_p^2}{\omega_0\gamma} = \left(\dfrac{\omega_p}{\omega_0}\right)^2\left(\dfrac{\omega_0}{\gamma}\right) \gg 1$, for a sharp resonance with $\gamma \ll \omega_0$ (and generally we also have $\omega_0 < \omega_p$).

In this region we have $\epsilon_1 \approx O(1)$, and so in region (2) we have $\boxed{\epsilon_2 \gg \epsilon_1}$. We can proceed similarly to what we did for regions (1) and (4), only now expanding the square roots of Eq. (5.2.8) for small ϵ_1/ϵ_2. We get,

$$k_1 \approx \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\epsilon_2\left[1+\frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2}\right)^2\right]+\frac{1}{2}\epsilon_1\right]^{1/2} = \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\epsilon_2+\frac{1}{4}\frac{\epsilon_1^2}{\epsilon_2}+\frac{1}{2}\epsilon_1\right]^{12} \quad (5.2.13)$$

$$= \frac{\omega}{c}\sqrt{\mu}\sqrt{\frac{\epsilon_2}{2}}\left[1+\frac{\epsilon_1}{\epsilon_2}+\frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2}\right)^2\right]^{1/2} \approx \frac{\omega}{c}\sqrt{\frac{\mu\epsilon_2}{2}}+\text{ small correction} \quad (5.2.14)$$

Similarly,

$$k_2 \approx \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\epsilon_2\left[1+\frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2}\right)^2\right]-\frac{1}{2}\epsilon_1\right]^{1/2} = \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\epsilon_2+\frac{1}{4}\frac{\epsilon_1^2}{\epsilon_2}-\frac{1}{2}\epsilon_1\right]^{1/2} \quad (5.2.15)$$

$$= \frac{\omega}{c}\sqrt{\mu}\sqrt{\frac{\epsilon_2}{2}}\left[1-\frac{\epsilon_1}{\epsilon_2}+\frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2}\right)^2\right]^{1/2} \approx \frac{\omega}{c}\sqrt{\frac{\mu\epsilon_2}{2}}+\text{ small correction} \quad (5.2.16)$$

And so in region (2) we have $\boxed{k_1 \approx k_2}$.

We could have gotten this more simply by saying that in region (2), since $\epsilon_1 \ll \epsilon_2$, then to lowest order we can take $\epsilon_1 \approx 0$, and so $\epsilon = i\epsilon_2$. Then $k = (\omega/c)\sqrt{\mu\epsilon} = (\omega/c)\sqrt{\mu\, i\epsilon_2} = (\omega/c)\sqrt{\mu\epsilon_2}\sqrt{i}$. Using $\sqrt{i} = (1+i)/\sqrt{2}$ then gives $k_1 = k_2 = (\omega/c)\sqrt{\mu\epsilon_2/2}$. The above, more involved, calculation lets one compute the corrections to this leading order result.

Since $k_1 \approx k_2$, within one wavelength of propagation into the material the wave amplitude has decayed by a factor $\mathrm{e}^{-2\pi k_2/k_1} \approx \mathrm{e}^{-2\pi} \approx 0.002$. The wave is very *strongly attenuated.*

Physically what is happening is the following. The wave excites atoms near their resonant frequency ω_0, which leads to large atomic displacements, which leads to large absorption of energy by the atomic damping force. The wave loses energy to the material and so the wave amplitude decays rapidly as the wave propagates into the material. Region (2) is the region of strong attenuation, or equivalently the region of resonant absorption. You should recall the same type of behavior from mechanics when you studied the damped harmonic oscillator. When the damped harmonic oscillator is driven by a force oscillating near the oscillator's natural frequency, then the amplitude of oscillation is largest, the phase of the displacement is $\pi/2$ out of phase with the force, and the absorption of energy is the greatest.

Region (3): $\boxed{\epsilon_1 < 0 \text{ and } |\epsilon_1| \gg \epsilon_2 \quad \Rightarrow \quad \text{Total Reflection}}$

The width of region (3) is $\omega_1 - \omega_0 = \sqrt{\omega_0^2+\omega_p^2} - \omega_0 \sim \omega_p \sim \sqrt{n}$, where n is the density of atoms in the dielectric. This follows since generally $\omega_0 \ll \omega_p$. Thus the width of this region increases as the material gets denser.

We can now compute k_1 and k_2 with similar expressions as we used in regions (1) and (4), only now we need to use $|\epsilon_1|$ when we factor it out of a square root. We have,

$$k_1 = \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\sqrt{\epsilon_1^2+\epsilon_2^2}+\frac{1}{2}\epsilon_1\right]^{1/2} = \frac{\omega}{c}\sqrt{\mu}\left[\frac{|\epsilon_1|}{2}\sqrt{1+\frac{\epsilon_2^2}{\epsilon_1^2}}+\frac{1}{2}\epsilon_1\right]^{1/2} \quad (5.2.17)$$

$$\approx \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}|\epsilon_1|+\frac{1}{4}\frac{\epsilon_2^2}{|\epsilon_1|}+\frac{1}{2}\epsilon_1\right]^{1/2} \qquad \text{since } |\epsilon_1| = -\epsilon_1\text{, those two terms cancel, giving} \quad (5.2.18)$$

$$= \frac{\omega}{c}\sqrt{\mu|\epsilon_1|}\,\frac{\epsilon_2}{2|\epsilon_1|} \quad (5.2.19)$$

Similarly,

$$k_2 = \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\sqrt{\epsilon_1^2+\epsilon_2^2}-\frac{1}{2}\epsilon_1\right]^{1/2} = \frac{\omega}{c}\sqrt{\mu}\left[\frac{|\epsilon_1|}{2}\sqrt{1+\frac{\epsilon_2^2}{\epsilon_1^2}}-\frac{1}{2}\epsilon_1\right]^{1/2} \tag{5.2.20}$$

$$\approx \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}|\epsilon_1|+\frac{1}{4}\frac{\epsilon_2^2}{|\epsilon_1|}-\frac{1}{2}\epsilon_1\right]^{1/2} \qquad \text{since } |\epsilon_1| = -\epsilon_1\text{, those two terms add, giving} \tag{5.2.21}$$

$$= \frac{\omega}{c}\sqrt{\mu|\epsilon_1|}\left[1+\frac{\epsilon_2^2}{8\epsilon_1^2}\right] = \frac{\omega}{c}\sqrt{\mu|\epsilon_1|} + \text{small correction} \tag{5.2.22}$$

So now $\boxed{\dfrac{k_2}{k_1} = \dfrac{2|\epsilon_1|}{\epsilon_2} \gg 1}$.

In one wavelength of propagation into the material the amplitude decays by a factor $\mathrm{e}^{-2\pi k_2/k_1}$ which becomes essentially zero when $k_2 \gg k_2$. The wave decays *much* more rapidly than in the region (2) of resonant absorption.

We could have gotten this result more simply by saying that, since $\epsilon_2 \ll |\epsilon_1|$ in region (3), then to lowest order $\epsilon_2 \approx 0$ and $\epsilon = \epsilon_1 = -|\epsilon_1|$. Then the dispersion relation gives $k = (\omega/c)\sqrt{\mu\epsilon} = (\omega/c)\sqrt{-\mu|\epsilon_1|} = i(\omega/c)\sqrt{\mu|\epsilon_1|}$. Thus k is pure imaginary, and the wave decays without any oscillations! The above more detailed calculation gives the corrections to this leading order behavior.

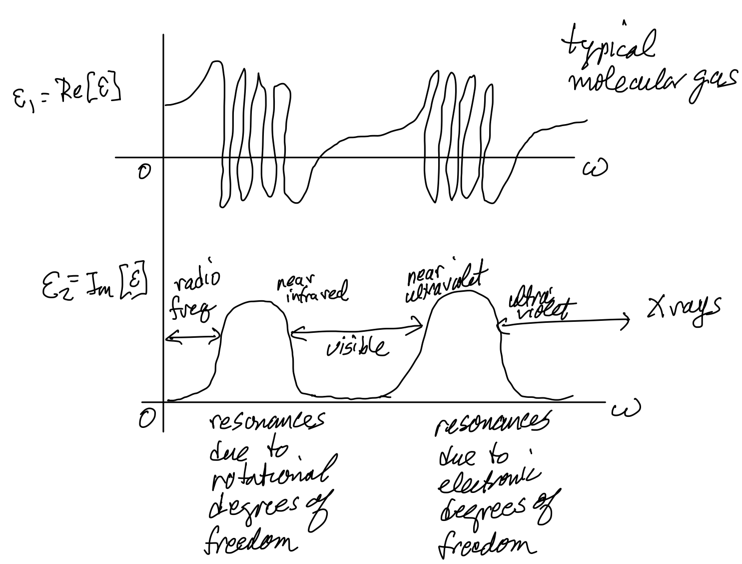
Since region (3) is well above the region of resonant absorption near ω_0, there is little energy being transferred from the wave to the material. Yet the amplitude of the wave decays dramatically as the wave tries to propagate into the material. This strong attenuation of the wave is due to the destructive interference between the wave and the induced fields of the polarized atoms, which are oscillating π out of phase with the driving electric field of the wave. We will see later that this corresponds to a *total reflection* of the wave.

More Realistic Materials

Our simple model for a polarizable atom had only a single resonance at ω_0. A more realistic model for molecule will have many bands of resonances due to the rotational, vibrational, and electronic modes of excitation of the molecule. In general we have,

$$\epsilon(\omega) = 1 + \omega_p^2 \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i} \tag{5.2.23}$$

where the $\hbar\omega_i$ are the spacings between the energy levels of the molecule with allowed electric dipole transitions, and the f_i are related to the matrix elements associated with those transitions.



In general, for a typical gas molecule, there are two bands of resonances, a low frequency band due to the rotational degrees of freedom of the molecule, and a high frequency band du e to the electronic degrees of freedom. The frequency spectrum is as sketched to the left. This spectrum suggests why organisms developed as as to see in the visible range of frequencies!

The Plasma Frequency

In the above notes we have commented that typically $\omega_0 \ll \omega_p$. Here we explore this claim.

We have

$$\omega_p \equiv \sqrt{\frac{4\pi ne^2}{m}} = 4.4 \times 10^{-16} \sqrt{\frac{n}{n_A}}\ \mathrm{sec}^{-1} \tag{5.2.24}$$

where n is the density of polarizable atoms/molecules, m is the mass of the electron, and $n_A = 6 \times 10^{23}/\mathrm{cm}^3$ is Avogadro's number of particles per cubic centimeter.

The corresponding energy is

$$\hbar\omega_p = 185 \sqrt{\frac{n}{n_A}}\ \mathrm{eV} \tag{5.2.25}$$

For water H_2O,

$$\frac{n}{n_A} \approx 0.05 \quad \Rightarrow \quad \hbar\omega_p \approx 40\ \mathrm{eV} \tag{5.2.26}$$

For a typical metal,

$$\frac{n}{n_A} \approx 0.1 \quad \Rightarrow \quad \hbar\omega_p \approx 58\ \mathrm{eV} \tag{5.2.27}$$

Compare that to the typical energy of electron level spacings,

$$\hbar\omega_0 \approx O(1)\ \mathrm{eV} \tag{5.2.28}$$

So indeed we generally have $\omega_0 \ll \omega_p$.

Summary

To summarize the results for *transverse* wave propagation:

When $\epsilon_1 > 0$ and $\epsilon_2 \ll \epsilon_1$, we are in a region of transparent propagation with $k_2 \ll k_1$.

When $\epsilon_2 \gg |\epsilon_1|$, we are in a region of resonant absorption with $k_1 \approx k_2$.

When $\epsilon_1 < 0$ and $\epsilon_2 \ll |\epsilon_1|$, we are in a region of total reflection with $k_2 \gg k_1$.

Notes for ϵ_1 and ϵ_2 in our simple model

Behavior of $\epsilon_1(\omega)$

location of maximum and minimum of ϵ_1

With $\epsilon_1 = 1 + \dfrac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$, the maximum and minimum are located by,

$$\frac{d\epsilon_1}{d\omega} = 0 \quad \Rightarrow \quad -2\omega[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2] - (\omega_0^2 - \omega^2)[2(\omega_0^2 - \omega^2)(-2\omega) + 2\omega\gamma^2] = 0 \tag{5.2.29}$$

Multiply out the terms to get

$$-2\omega(\omega_0^2 - \omega^2)^2 - 2\omega^3\gamma^2 + 4\omega(\omega_0^2 - \omega^2)^2 - (\omega_0^2 - \omega^2)2\omega\gamma^2 = 0 \tag{5.2.30}$$

$$2\omega(\omega_0^2 - \omega^2)^2 - 2\omega^3\gamma^2 - 2\omega_0^2\omega\gamma^2 + 2\omega^3\gamma^2 = 0 \tag{5.2.31}$$

The 2nd and 4th terms cancel, then divide each term by 2ω to get,

$$(\omega_0^2 - \omega^2)^2 - \omega_0^2\gamma^2 = 0 \quad \Rightarrow \quad \omega_0^2 - \omega^2 = \pm\omega_0\gamma \quad \Rightarrow \quad \omega^2 = \omega_0^2 \mp \omega_0\gamma \tag{5.2.32}$$

So

$$\omega \;=\; \sqrt{\omega_0^2 \mp \omega_0\gamma} \;=\; \omega_0\sqrt{1 \mp \frac{\gamma}{\omega_0}} \;\approx\; \omega_0\left(1 \mp \frac{\gamma}{2\omega_0}\right) \qquad \text{for a sharp resonance with } \gamma/\omega_0 \ll 1 \tag{5.2.33}$$

So

$$\omega = \omega_0 \mp \frac{\gamma}{2} \qquad \text{give the locations of the minimum and maximum of } \epsilon_1 \tag{5.2.34}$$

location of the zeros of ϵ_1

The zeros of ϵ_1 are determined by,

$$\epsilon_1 = 0 \quad \Rightarrow \quad (\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2 + \omega_p^2(\omega_0^2 - \omega^2) = 0 \tag{5.2.35}$$

$$\Rightarrow \quad \omega^4 - 2\left(\omega_0^2 + \frac{\omega_p^2}{2} - \frac{\gamma^2}{3}\right)\omega^2 + \omega_0^4 + \omega_p^2\omega_0^2 = 0 \tag{5.2.36}$$

We can solve the quadratic equation to get,

$$\omega^2 = \omega_0^2 + \frac{\omega_p^2}{2} - \frac{\gamma^2}{2} \pm \sqrt{\frac{\omega_p^4}{4} + \frac{\gamma^4}{4} - \omega_0^2\gamma^2 - \frac{\omega_p^2\gamma^2}{2}} \tag{5.2.37}$$

Consider the zero at the larger ω_1 shown in the figure. This is the $(+)$ root. When $\gamma \ll \omega_0$, this is far from ω_0 on the scale of γ, so to leading order we can ignore all the terms involving γ. We then get

$$\omega_1^2 = \omega_0^2 + \frac{\omega_p^2}{2} + \frac{\omega_p^2}{2} = \omega_0^2 + \omega_p^2 \quad \Rightarrow \quad \omega_1 = \sqrt{\omega_0^2 + \omega_p^2} \tag{5.2.38}$$

For the zero between the maximum and minimum, we take the (−) root and keep the lowest terms $\sim O(\gamma^2)$. We get,

$$\omega^2 = \omega_0^2 + \frac{\omega_p^2}{2} - \frac{\gamma^2}{2} - \frac{\omega_p^2}{2}\sqrt{1 - \frac{4\omega_0^2\gamma^2}{\omega_p^4} - \frac{2\gamma^2}{\omega_p^2}} \quad \approx \quad \omega_0^2 + \frac{\omega_p^2}{2} - \frac{\gamma^2}{2} - \frac{\omega_p^2}{2}\left(1 - \frac{2\omega_0^2\gamma^2}{\omega_p^4} - \frac{\gamma^2}{\omega_p^2}\right) \tag{5.2.39}$$

$$= \omega_0^2 + \frac{\omega_p^2}{2} - \frac{\gamma^2}{2} - \frac{\omega_p^2}{2} + \frac{\omega_0^2\gamma^2}{\omega_p^2} + \frac{\gamma^2}{2} \quad = \quad \omega_0^2\left(1 + \frac{\gamma^2}{\omega_p^2}\right) \tag{5.2.40}$$

$$\Rightarrow \quad \omega = \omega_0\sqrt{1 + \frac{\gamma^2}{\omega_p^2}} \quad \approx \quad \omega_0\left(1 + \frac{\gamma^2}{2\omega_p^2}\right) \quad = \quad \omega_0 + \frac{\omega_0\gamma^2}{2\omega_p^2} \quad = \quad \omega_0 + \frac{1}{2}\left(\frac{\omega_0^2}{\omega_p^2}\right)\left(\frac{\gamma^2}{\omega_0^2}\right) \tag{5.2.41}$$

So the zero is shifted upwards a bit from ω_0, as is obvious in the figure. Since $\gamma \ll \omega_0$, and usually $\omega_0 \ll \omega_p$, this shift is very small.

<u>Behavior of $\epsilon_2(\omega)$</u>

location of the peak

With $\epsilon_2 = \dfrac{\omega_p^2\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$, the peak is located by,

$$\frac{d\epsilon_2}{d\omega} = 0 \quad \Rightarrow \quad [(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2] - \omega[2(\omega_0^2 - \omega^2)(-2\omega) + 2\omega\gamma^2] = 0 \tag{5.2.42}$$

$$\Rightarrow \quad \omega^4 + 2\left(\frac{\gamma^2}{6} - \frac{\omega_0^2}{3}\right)\omega^2 - \frac{\omega_0^4}{3} = 0 \qquad \text{quadratic equation for } \omega^2 \tag{5.2.43}$$

$$\Rightarrow \quad \omega^2 = \frac{\omega_0^2}{3} - \frac{\gamma^2}{6} + \sqrt{\left(\frac{\gamma^2}{6} - \frac{\omega_0^2}{3}\right)^2 + \frac{\omega_0^4}{3}} \tag{5.2.44}$$

$$\Rightarrow \quad \omega^2 = \frac{\omega_0^2}{3} - \frac{\gamma^2}{6} + \frac{2\omega_0^2}{3}\sqrt{1 - \frac{\gamma^2}{4\omega_0^2} + \frac{\gamma^4}{16\omega_0^4}} \tag{5.2.45}$$

For a sharp resonance with $\gamma \ll \omega_0$, we can neglect the $(\gamma/\omega_0)^4$ term in the square root compared to the $(\gamma/\omega_0)^2$ term, and then expand the square root to lowest order to get

$$\omega^2 = \frac{\omega_0^2}{3} - \frac{\gamma^2}{6} + \frac{2\omega_0^2}{3}\left(1 - \frac{\gamma^2}{8\omega_0^2}\right) = \omega_0^2 - \frac{\gamma^2}{4} = \bar{\omega}^2 \qquad \text{defined earlier in Eq. (5.1.25)} \tag{5.2.46}$$

height of the peak

The peak value of ϵ_2 is then

$$(\epsilon_2)_{\max} = \frac{\omega_p^2\bar{\omega}\gamma}{(\omega_0^2 - \bar{\omega}^2)^2 + \bar{\omega}^2\gamma^2} = \frac{\omega_p^2\bar{\omega}\gamma}{(\gamma^2/4)^2 + \bar{\omega}^2\gamma^2} \approx \frac{\omega_p^2}{\bar{\omega}\gamma} \approx \frac{\omega_p^2}{\omega_0\gamma} \qquad \text{when } \gamma \ll \omega_0 \tag{5.2.47}$$

width of the peak

The frequency ω^* where the peak in ϵ_2 drops to half its height is when

$$\epsilon_2(\omega^*) = \frac{\omega_p^2\omega^*\gamma}{(\omega_0^2 - \omega^{*2})^2 + \omega^{*2}\gamma^2} = \frac{\omega_p^2}{2\bar{\omega}\gamma} \tag{5.2.48}$$

For $\gamma \ll \omega_0$, we can take to lowest order, $\bar{\omega} \approx \omega^* \approx \omega_0$, to write $(\omega_0^2 - {\omega^*}^2) = (\omega_0 - \omega^*)(\omega_0 + \omega^*) \approx \Delta\omega(2\omega_0)$, with $\Delta\omega = \omega_0 - \omega^*$. Then

$$\epsilon_2(\omega^*) = \frac{\omega_p^2 \omega_0 \gamma}{4\omega_0^2(\Delta\omega)^2 + \omega_0^2\gamma^2} = \frac{\omega_p^2}{2\omega_0\gamma} \quad \Rightarrow \quad \frac{(\omega_0\gamma)^2}{4\omega_0^2(\Delta\omega)^2 + (\omega_0\gamma)^2} = \frac{1}{2} \quad \Rightarrow \quad (\Delta\omega)^2 = \frac{\gamma^2}{4} \quad \Rightarrow \quad \omega^* = \omega_0 \pm \frac{\gamma}{2} \tag{5.2.49}$$

So the peak in ϵ_2 has a width at half height of γ.