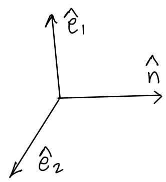
Unit 5-4: Polarization of EM Waves

In this section we will discuss the polarization of electromagnetic waves. Here *polarization* refers to the directional orientation of the $\mathbf{E}$ and $\mathbf{B}$ fields of the wave (so don't confuse it with the different meaning of "polarization" when we are talking about polarizable atoms!).

Consider a transverse plane wave traveling in the $\hat{\mathbf{n}}$ direction, i.e., $\mathbf{k} = k\hat{\mathbf{n}}$. We define a *right handed* coordinate system as follows:



$$\begin{aligned} \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 &= \hat{\mathbf{n}} \\ \hat{\mathbf{n}} \times \hat{\mathbf{e}}_1 &= \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_2 \times \hat{\mathbf{n}} &= \hat{\mathbf{e}}_1 \end{aligned} \tag{5.4.1}$$

A general solution for Maxwell's equations for a transverse plane wave is then,

$$\mathbf{E}(\mathbf{r},t) = \text{Re}\left[(E_1\hat{\mathbf{e}}_1 + E_2\hat{\mathbf{e}}_2)\,\text{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] \tag{5.4.2}$$

$$\mathbf{H}(\mathbf{r},t) = \frac{c}{\omega\mu}\text{Re}\left[k\hat{\mathbf{n}} \times (E_1\hat{\mathbf{e}}_1 + E_2\hat{\mathbf{e}}_2)\,\text{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] = \frac{c}{\omega\mu}\text{Re}\left[k(E_1\hat{\mathbf{e}}_2 - E_2\hat{\mathbf{e}}_1)\,\text{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] \tag{5.4.3}$$

In general, k is complex, $k = k_1 + ik_2 = |k|\text{e}^{i\delta}$, with $|k| = \sqrt{k_1^2 + k_2^2}$ and $\delta = \arctan(k_2/k_1)$.

So far we implicitly assumed that E_1 and E_2 were *real* valued constants. In this case, when we take the real part of the complex exponential we get,

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_\omega \text{e}^{-k_2\hat{\mathbf{n}}\cdot\mathbf{r}}\cos(k_1\hat{\mathbf{n}}\cdot\mathbf{r} - \omega t) \tag{5.4.4}$$

$$\mathbf{H}(\mathbf{r},t) = \mathbf{H}_\omega \text{e}^{-k_2\hat{\mathbf{n}}\cdot\mathbf{r}}\cos(k_1\hat{\mathbf{n}}\cdot\mathbf{r} - \omega t + \delta) \tag{5.4.5}$$

where $\mathbf{E}_\omega = E_1\hat{\mathbf{e}}_1 + E_2\hat{\mathbf{e}}_2$ and $H_\omega = \dfrac{c|k|}{\omega\mu}(E_1\hat{\mathbf{e}}_2 - E_2\hat{\mathbf{e}}_1)$ are *fixed* constant vectors.

In this case the *directions* of $\mathbf{E}$ and $\mathbf{H}$ remain the same at all points in space and at all times, while the amplitudes oscillate in time and space. Such a plane wave is called a *linearly* polarized wave.

However there is nothing to prevent one from choosing a solution with E_1 and E_2 as complex valued numbers,

$$E_1 = |E_1|\text{e}^{i\chi_1} \qquad \text{and} \qquad E_2 = |E_2|\text{e}^{i\chi_2} \tag{5.4.6}$$

In this case one has,

$$\mathbf{E}(\mathbf{r},t) = \text{Re}\left[|E_1|\,\hat{\mathbf{e}}_1\,\text{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t+\chi_1)} + |E_2|\,\hat{\mathbf{e}}_2\,\text{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t+\chi_2)}\right] \tag{5.4.7}$$

$$= \text{e}^{-k_z\hat{\mathbf{n}}\cdot\mathbf{r}}\left[|E_1|\,\hat{\mathbf{e}}_1\,\cos(k_1\hat{\mathbf{n}}\cdot\mathbf{r} - \omega t + \chi_1) + |E_2|\,\hat{\mathbf{e}}_2\,\cos(k_1\hat{\mathbf{n}}\cdot\mathbf{r} - \omega t + \chi_2)\right] \tag{5.4.8}$$

and

$$\mathbf{H}(\mathbf{r},t) = \frac{c|k|}{\omega\mu}\text{Re}\left[|E_1|\,\hat{\mathbf{e}}_2\,\text{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta+\chi_1)} - |E_2|\,\hat{\mathbf{e}}_1\,\text{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta+\chi_2)}\right] \tag{5.4.9}$$

$$= \frac{c|k|}{\omega\mu}\text{e}^{-k_2\hat{\mathbf{n}}\cdot\mathbf{r}}\left[|E_1|\,\hat{\mathbf{e}}_2\,\cos(k_1\hat{\mathbf{n}}\cdot\mathbf{r} - \omega t + \delta + \chi_1) - |E_2|\,\hat{\mathbf{e}}_1\,\cos(k_1\hat{\mathbf{n}}\cdot\mathbf{r} - \omega t + \delta + \chi_2)\right] \tag{5.4.10}$$

Unless $\chi_1 = \chi_2$ we see that the components of $\mathbf{E}$ and $\mathbf{H}$ in directions $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ will oscillate out of phase with each other. Thus, not only will the amplitudes of $\mathbf{E}$ and $\mathbf{H}$ oscillate in time and space, but also the *directions* of $\mathbf{E}$ and

H will oscillate in time and space. The directions of **E** and **H** are no longer fixed. We will see that this situation in general corresponds to an *elliptically polarized* wave.

General Case

E_1 and E_2 are complex constants. Write

$$E_1\hat{\mathbf{e}}_1 + E_2\hat{\mathbf{e}}_2 = \mathbf{U}e^{i\psi} \qquad \text{where } \psi \text{ is chosen so that } \mathbf{U}\cdot\mathbf{U} \text{ is } \textit{real} \text{ valued.} \tag{5.4.11}$$

One can always do this since

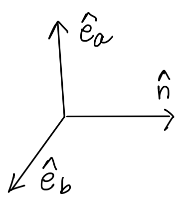
$$\mathbf{U}\cdot\mathbf{U} = (E_1^2 + E_2^2)e^{-2i\psi} \qquad \text{so choose } 2\psi \text{ to be the phase of the complex } E_1^2 + E_2^2. \tag{5.4.12}$$

U is a complex valued vector, we can write $\mathbf{U} = \mathbf{U}_a + i\mathbf{U}_b$, where $\mathbf{U}_a$ and $\mathbf{U}_b$ are *real valued* vectors. Then

$$\mathbf{U}\cdot\mathbf{U} \ \text{ is real valued} \quad \Rightarrow \quad \mathrm{Im}\left[(\mathbf{U}_a + i\mathbf{U}_b)\cdot(\mathbf{U}_a + i\mathbf{U}_b)\right] = 0 \quad \Rightarrow \quad \mathbf{U}_a\cdot\mathbf{U}_b = 0 \quad \text{so} \quad \mathbf{U}_a \perp \mathbf{U}_b \tag{5.4.13}$$

Let $\hat{\mathbf{e}}_a$ be the unit vector in the direction of $\mathbf{U}_a$, so that $\mathbf{U}_a = U_a\hat{\mathbf{e}}_a$ where $U_a = |\mathbf{U}_a|$.

Let $\hat{\mathbf{e}}_b = \hat{\mathbf{n}}\times\hat{\mathbf{e}}_a$ so that $\{\hat{\mathbf{n}}, \hat{\mathbf{e}}_a, \hat{\mathbf{e}}_b\}$ form a right handed coordinate system.



Then $\mathbf{U}_b = \pm U_b\hat{\mathbf{e}}_b$ where $U_b = |\mathbf{U}_b|$, since $\mathbf{U}_a \perp \mathbf{U}_b$ and both $\mathbf{U}_a$ and $\mathbf{U}_b$ are perpendicular to $\hat{\mathbf{n}}$.

The sign in $\mathbf{U}_b = \pm U_b\hat{\mathbf{e}}_b$ is $(+)$ if $\mathbf{U}_b$ is parallel to $\hat{\mathbf{e}}_b$, and it is $(-)$ if $\mathbf{U}_b$ is antiparallel to $\hat{\mathbf{e}}_b$.

In this representation we have,

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{U}e^{i\psi}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] = \mathrm{Re}\left[(\mathbf{U}_a + i\mathbf{U}_b)\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t+\psi)}\right] \tag{5.4.14}$$

$$= e^{-k_2\hat{\mathbf{n}}\cdot\mathbf{r}}\,\mathrm{Re}\left[U_a\,\hat{\mathbf{e}}_a\,e^{i(k_1\hat{\mathbf{n}}\cdot\mathbf{r}-\omega t+\psi)} \pm iU_b\,\hat{\mathbf{e}}_b\,e^{i(k_1\hat{\mathbf{n}}\cdot\mathbf{r}-\omega t+\psi)}\right] \tag{5.4.15}$$

$$= e^{-k_2\hat{\mathbf{n}}\cdot\mathbf{r}}\left[U_a\,\hat{\mathbf{e}}_a\,\cos(\Phi+\psi) \mp U_b\,\hat{\mathbf{e}}_b\,\sin(\Phi+\psi)\right] \qquad \text{where} \quad \Phi \equiv k_1\hat{\mathbf{n}}\cdot\mathbf{r} - \omega t \tag{5.4.16}$$

Let us define $\tilde{U}_a \equiv e^{-k_2\hat{\mathbf{n}}\cdot\mathbf{r}}\,U_a$ and $\tilde{U}_b \equiv e^{-k_2\hat{\mathbf{n}}\cdot\mathbf{r}}\,U_b$.

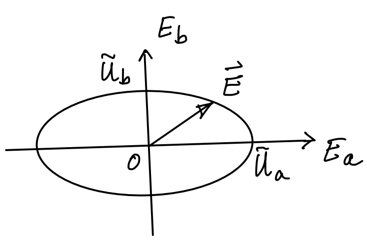
Then define E_a and E_b as the components of **E** in the directions $\hat{\mathbf{e}}_a$ and $\hat{\mathbf{e}}_b$, respectively.

$$E_a = \tilde{U}_a\cos(\Phi+\psi) \qquad \text{and} \qquad E_b = \mp\tilde{U}_b\sin(\Phi+\psi) \tag{5.4.17}$$

This then gives,

$$\left(\frac{E_a}{\tilde{U}_a}\right)^2 + \left(\frac{E_b}{\tilde{U}_b}\right)^2 = \cos^2(\Phi+\psi) + \sin^2(\Phi+\psi) = 1 \tag{5.4.18}$$

which is just the equation for an ellipse with semi-axes of length $\tilde{U}_a$ and $\tilde{U}_b$, oriented in the directions of $\hat{\mathbf{e}}_a$ and $\hat{\mathbf{e}}_b$.



At a fixed position in space **r**, the tip of the vector **E** will trace out the ellipse as the time increases by one period of oscillation, $2\pi/\omega$.

For $(+)$, i.e., $\mathbf{U}_b = +U_b\hat{\mathbf{e}}_b$, the tip of **E** goes around the ellipse *counterclockwise* as t increases.

For $(-)$, i.e., $\mathbf{U}_b = -U_b\hat{\mathbf{e}}_b$, the tip of **E** goes around the ellipse *clockwise* as t increases.

Such a wave is said to be *elliptically polarized.*

Special Cases

1) If $U_a = 0$ or $U_b = 0$ the wave is *linearly polarized.* The direction of $\mathbf{E}$ stays fixed, while the amplitude of $\mathbf{E}$ oscillates.

2) If $U_a = U_b$ then the tip of $\mathbf{E}$ traces out a circle as t increases. The wave is said to be *circularly polarized.* The $(+)$ case is said to be *right handed circularly polarized*, while the $(-)$ case is said to be *left handed circularly polarized.*

One can define circular polarization basis vectors,

$$\hat{\mathbf{e}}_+ \equiv \frac{\hat{\mathbf{e}}_a + i\hat{\mathbf{e}}_b}{\sqrt{2}} \qquad \text{and} \qquad \hat{\mathbf{e}}_- = \frac{\hat{\mathbf{e}}_a - i\hat{\mathbf{e}}_b}{\sqrt{2}} \qquad \text{where } \hat{\mathbf{e}}_a \perp \hat{\mathbf{e}}_b. \tag{5.4.19}$$

A wave with complex amplitude $\mathbf{E}_\omega = E\,\hat{\mathbf{e}}_+$ is right handed circularly polarizied.

A wave with complex amplitude $\mathbf{E}_\omega = E\,\hat{\mathbf{e}}_-$ is left handed circularly polarized.

Just as the general case can always be written as a superposition of two orthogonal linearly polarized waves,

$$\mathbf{E}_\omega = E_1\hat{\mathbf{e}}_1 + E_2\hat{\mathbf{e}}_2 \tag{5.4.20}$$

one can also always write the general case as a superposition of a right handed and a left handed circularly polarized waves,

$$\mathbf{U} = \mathbf{U}_a + i\mathbf{U}_b = U_a\hat{\mathbf{e}}_a \pm iU_b\hat{\mathbf{e}}_b = \left(\frac{U_a + U_b}{\sqrt{2}}\right)\hat{\mathbf{e}}_\pm + \left(\frac{U_a - U_b}{\sqrt{2}}\right)\hat{\mathbf{e}}_\mp \tag{5.4.21}$$

(substitute in for $\hat{\mathbf{e}}_\pm$ and expand the expression to see that this is so).

So an elliptically polarized wave can be written as a superposition of circularly polarized waves.

As a special case of the above (if $U_a = 0$ or $U_b = 0$), a *linearly* polarized wave can always be written as a superposition of a right handed and a left handed circularly polarized wave.