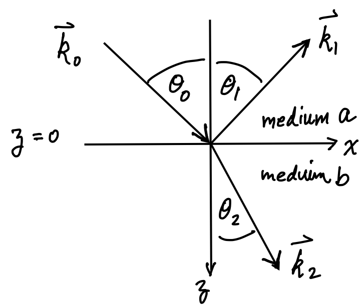
Unit 5-5: Reflection and Transmission of EM Waves at Interfaces

Consider an EM wave propagating from medium a into medium b. For simplicity we assume that ϵ_a is real and positive (so medium a is transparent), while ϵ_b may be complex. We take μ_a and μ_b to be real and constant.



$\mathbf{k}_0$ is the incident wave — θ_0 is the angle of incidence

$\mathbf{k}_1$ is the reflected wave — θ_1 is the angle of reflection

$\mathbf{k}_2$ is the transmitted, or *refracted*, wave — θ_2 is the angle of transmission

Snell's Law

Let each wave be given by

$$\mathbf{F}_n(\mathbf{r},t) = \mathbf{F}_n \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \text{where } \mathbf{F}_n \text{ can be either } \mathbf{E}_n \text{ or } \mathbf{H}_n \text{ for the electric or magnetic part of the wave} \tag{5.5.1}$$

We apply the boundary conditions at the interface between medium a and medium b. The tangential component of $\mathbf{E}$ is continuous at the interface at $z=0$. If $\hat{\mathbf{t}}$ is any unit vector in the xy plane, and we consider the position $\mathbf{r}=0$, then equating the tangential component of $\mathbf{E}$ in medium a with the tangential component of $\mathbf{E}$ in medium b, this boundary condition becomes,

$$\hat{\mathbf{t}}\cdot\mathbf{E}_0\,\mathrm{e}^{-i\omega_0 t} + \hat{\mathbf{t}}\cdot\mathbf{E}_1\,\mathrm{e}^{-i\omega_1 t} = \hat{\mathbf{t}}\cdot\mathbf{E}_2\,\mathrm{e}^{-i\omega_2 t} \tag{5.5.2}$$

This must hold true for all times t. The only way that can happen is if all terms oscillate at the same frequency, i.e.

$$\boxed{\omega_0 = \omega_1 = \omega_2 \equiv \omega} \tag{5.5.3}$$

Now consider the same boundary condition for a position vector $\mathbf{r}_\perp$ that lies in the xy plane at $z=0$. Since all the ω's are equal, we can cancel out the common $\mathrm{e}^{-i\omega t}$ factors to get,

$$\hat{\mathbf{t}}\cdot\mathbf{E}_0\,\mathrm{e}^{i\mathbf{k}_0\cdot\mathbf{r}_\perp} + \hat{\mathbf{t}}\cdot\mathbf{E}_1\,\mathrm{e}^{i\mathbf{k}_1\cdot\mathbf{r}_\perp} = \hat{\mathbf{t}}\cdot\mathbf{E}_2\,\mathrm{e}^{i\mathbf{k}_2\cdot\mathbf{r}_\perp} \tag{5.5.4}$$

This must hold true for all $\mathbf{r}_\perp$ in the xy plane at $z=0$. This can only happen if the projections of the $\mathbf{k}_n$ in the xy plane are all equal (this boundary condition places no constraint on the z components of the $\mathbf{k}_n$ since $\hat{\mathbf{z}}\cdot\mathbf{r}_\perp = 0$),

$$\boxed{k_{0x} = k_{1x} = k_{2x} \qquad\qquad k_{0y} = k_{1y} = k_{2y}} \tag{5.5.5}$$

This means that the projections of the $\mathbf{k}_n$ into the xy plane are all colinear. Choose the coordinate system as in the diagram above so that all the $\mathbf{k}_n$ vectors lie in the xz plane ($\hat{\mathbf{y}}$ is pointing out of the page). This plane, in which $\mathbf{k}_0$, $\mathbf{k}_1$, and $\mathbf{k}_2$ all lie, is called the *plane of incidence*.

Since ϵ_a is by assumption real and positive, $\mathbf{k}_0$ and $\mathbf{k}_1$ are real valued vectors.

$$k_{0x} = k_{1x} \quad\Rightarrow\quad |\mathbf{k}_0|\sin\theta_0 = |\mathbf{k}_1|\sin\theta_1 \tag{5.5.6}$$

since the dispersion relation in medium a gives

$$k_0^2 = \frac{\omega^2}{c^2}\mu_a\epsilon_a \quad \text{and} \quad k_1^2 = \frac{\omega^2}{c^2}\mu_a\epsilon_a \tag{5.5.7}$$

then $|\mathbf{k}_0| = |\mathbf{k}_1|$ and so,

$$\sin\theta_0 = \sin\theta_1 \qquad \Rightarrow \qquad \boxed{\theta_0 = \theta_1} \qquad \underline{\text{the angle of reflection equals the angle of incidence}} \tag{5.5.8}$$

If ϵ_b is also real and positive (medium b is also transparent) then $\mathbf{k}_2$ is also a real valued vector and so,

$$k_{0x} = k_{2x} \quad \Rightarrow \quad |\mathbf{k}_0| \sin\theta_0 = |\mathbf{k}_2| \sin\theta_2 \tag{5.5.9}$$

From the dispersion relations in the two media,

$$k_0^2 = \frac{\omega^2}{c^2}\mu_a\epsilon_a \quad \text{and} \quad k_2^2 = \frac{\omega^2}{c^2}\mu_b\epsilon_b \tag{5.5.10}$$

the condition of Eq. (5.5.9) becomes,

$$\sqrt{\mu_a\epsilon_a}\sin\theta_0 = \sqrt{\mu_b\epsilon_b}\sin\theta_2 \tag{5.5.11}$$

In terms of the index of refraction $n \equiv \dfrac{kc}{\omega} = \dfrac{\omega\sqrt{\mu\epsilon}}{c}\dfrac{c}{\omega} = \sqrt{\mu\epsilon}$ we have,

$$n_a \sin\theta_0 = n_b \sin\theta_2 \qquad \Rightarrow \qquad \boxed{\frac{\sin\theta_2}{\sin\theta_0} = \frac{n_a}{n_b}} \qquad \underline{\text{Snell's Law}} \tag{5.5.12}$$

Both the above laws, (i) angle of incidence = angle of reflection, and (ii) Snell's Law, hold true for all types of waves, not just EM waves. This is because any sort of wave will necessarily involve some boundary condition at the interface, and then the matching of the oscillatory parts of the wave on either side of the interface will lead to these same laws.

If $n_a > n_b$, then $\theta_2 > \theta_0$. In this case, when θ_0 is too large we will have $\dfrac{n_a}{n_b}\sin\theta_0 > 1$ and there will be no solution for $\theta_2 \quad \Rightarrow \quad$ there will be no transmitted wave. When this happens it is known as *total internal reflection*. The wave does not exit medium a.

The critical angle, above which one has total internal reflection, is called the *critical angle* θ_c, and is given by,

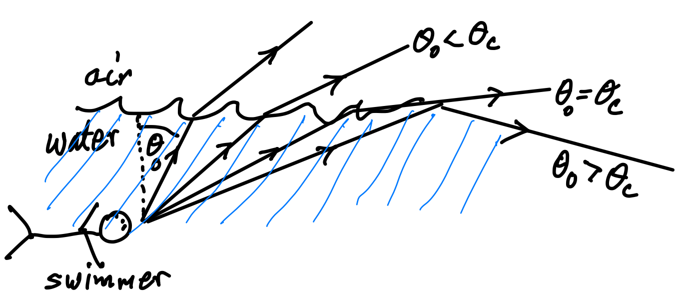
$$\frac{n_a}{n_b}\sin\theta_c = 1 \qquad \Rightarrow \qquad \boxed{\theta_c = \arcsin\left(\frac{n_b}{n_a}\right)} \tag{5.5.13}$$

Since $n = \sqrt{\mu\epsilon}$ and $\epsilon \approx 1 + 4\pi N\alpha$ (where N is the density of polarizable atoms), then the index of refraction n increases as the density of the material increases. One usually has total internal reflection when one goes from a denser to a less dense medium.

Examples:

Diamonds sparkle due to total internal reflection! Diamonds have large n and so a small θ_c. When cut properly, the light inside bounces around inside having many total internal reflections before it can escape.

You can also see total internal reflection directly for yourself if you go swimming under water. When you look up, if the angle of your gaze with respect the the surface is $\theta_0 < \theta_c$, then you will see out of the pool towards the ceiling. But if the angle of your gaze is at $\theta_0 = \theta_c$, your gaze will run along the surface of the water. And if the angle of your gaze is at $\theta_0 > \theta_c$, you won't see outside the water at all – your gaze will be reflected downwards back into the water.



Snell's Law for a Non-Transparent Medium

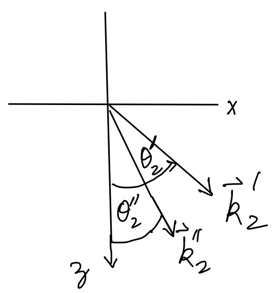
Now we consider the more general case where $\sqrt{\epsilon_b}$ can be complex valued, so $\mathbf{k}_2$ is a complex valued vector. We can write,

$$\mathbf{k}_2 = \mathbf{k}_2' + i\mathbf{k}_2'' \qquad \mathbf{k}_2' \text{ and } \mathbf{k}_2'' \text{ are real valued, and } k_2' \equiv |\mathbf{k}_2'| \text{ and } k_2'' = |\mathbf{k}_2''| \tag{5.5.14}$$

Note, the real part $\mathbf{k}_2'$ and the imaginary part $\mathbf{k}_2''$ do *not* need to be in the same direction!

Using the condition $k_{0x} = k_{2x}$, and noting that k_{0x} is real valued while k_{2x} is complex valued, we equate the real and imaginary parts to get,

$$k_{0x} = k_{2x}' \qquad \text{and} \qquad 0 = k_{2x}'' \tag{5.5.15}$$



If θ_2' is the angle that $\mathbf{k}_2'$ makes with respect to $\hat{\mathbf{z}}$, and θ_2'' is the angle that $\mathbf{k}_2''$ makes with respect to $\hat{\mathbf{z}}$, then the above gives,

$$k_0 \sin\theta_0 = k_2' \sin\theta_2' \qquad \text{and} \qquad 0 = k_2'' \sin\theta_2'' \quad \Rightarrow \quad \boxed{\theta_2'' = 0} \tag{5.5.16}$$

So $\theta_2'' = 0$ and $\mathbf{k}_2'' = k_2''\hat{\mathbf{z}}$. The attenuation factor of the transmitted wave is

$$\mathrm{e}^{-\mathbf{k}_2''\cdot\mathbf{r}} = \mathrm{e}^{-k_2'' z} \tag{5.5.17}$$

Thus we see that the planes of constant *amplitude* of the transmitted wave are always parallel to the plane of the interface, no matter what the angle of incidence θ_0 is. In contrast, the planes of constant *phase* of the transmitted wave will be orthogonal to $\mathbf{k}_2'$.

Having found θ_2'' there are still three quantities we must solve for in order to characterize the transmitted wave. These are θ_2', and the amplitudes k_2' and k_2''.

To solve for these we will need three equations. The first is:

$$1) \quad k_0 \sin\theta_0 = k_2' \sin\theta_2' \qquad \text{from the boundary condition} \tag{5.5.18}$$

The other two come from equating the real and imaginary parts of the dispersion relation in medium b.

$$k_2^2 = \frac{\omega^2}{c^2}\mu_b\epsilon_b = \frac{\omega^2}{c^2}\mu_b(\epsilon_{b1} + i\epsilon_{b2}) \tag{5.5.19}$$

Now,

$$k_2^2 \;=\; (\mathbf{k}_2' + i\mathbf{k}_2'')\cdot(\mathbf{k}_2' + i\mathbf{k}_2'') \;=\; (k_2')^2 - (k_2'')^2 + 2i\mathbf{k}_2'\cdot\mathbf{k}_2'' \;=\; (k_2')^2 - (k_2'')^2 + 2ik_2'k_2''\cos\theta_2' \tag{5.5.20}$$

Use this to equate the real and imaginary parts of Eq. (5.5.19) and get,

$$2) \quad (k_2')^2 - (k_2'')^2 = \frac{\omega^2}{c^2}\mu_b\epsilon_{b1} \tag{5.5.21}$$

$$3) \quad 2k_2'k_2'' = \frac{\omega^2}{c^2}\mu_b\epsilon_{b2}\cos\theta_2' \tag{5.5.22}$$

We will use (2) and (3) to solve for k_2' and k_2'' in terms of θ_2'.

$$2) \quad \Rightarrow \quad (k_2')^2 = (k_2'')^2 + \frac{\omega^2}{c^2}\mu_b\epsilon_{b1} \tag{5.5.23}$$

$$3) \quad \Rightarrow \quad k_2'' = \frac{\omega^2}{c^2}\frac{\mu_b\epsilon_{b2}}{2k_2'\cos\theta_2'} \tag{5.5.24}$$

Plug Eq. (5.5.24) into (5.5.23) to get

$$(k_2')^2 = \left(\frac{\omega^2}{c^2}\,\frac{\mu_b\epsilon_{b2}}{2k_2'\cos\theta_2'}\right)^2 + \frac{\omega^2}{c^2}\mu_b\epsilon_{b1} \tag{5.5.25}$$

$$\Rightarrow \quad (k_2')^4 - \frac{\omega^2}{c^2}\,\mu_b\epsilon_{b1}(k_2')^2 - \frac{\omega^4}{c^4}\,\frac{\mu_b^2\epsilon_{b2}^2}{4\cos^2\theta_2'} = 0 \tag{5.5.26}$$

This is just a quadratic equation for $(k_2')^2$. We can solve using the quadratic formula.

$$(k_2')^2 = \frac{\omega^2\mu_b\epsilon_{b1}}{2c^2} + \sqrt{\frac{\omega^4\mu_b^2\epsilon_{b1}^2}{4c^4} + \frac{\omega^4\mu_b^2\epsilon_{b2}^2}{4c^4\cos^2\theta_2'}} \tag{5.5.27}$$

where we take the $(+)$ sign in the quadratic formula since $(k_2')^2$ must be positive.

$$(k_2')^2 = \frac{\omega^2\mu_b}{c^2}\left[\frac{\epsilon_{b1}}{2} + \frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_2'}}\,\right] \tag{5.5.28}$$

So finally,

$$k_2' = \frac{\omega}{c}\sqrt{\mu_b}\left[\frac{\epsilon_{b1}}{2} + \frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_2'}}\,\right]^{1/2} \tag{5.5.29}$$

and then we can get k_2'' from Eq. (5.5.24),

$$k_2'' = \frac{\omega}{c}\sqrt{\mu_b}\left[-\frac{\epsilon_{b1}}{2} + \frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_2'}}\,\right]^{1/2} \tag{5.5.30}$$

Note, these reduce to Eq. (5.2.8) that we found earlier for a plane wave in a medium with complex ϵ, if we take $\theta_2' = 0$. We will have $\theta_2' = 0$ for normal incidence $\theta_0 = 0$.

Both k_2' and k_2'' in Eqs. (5.5.29) and (5.5.30) above still depend on the angle of refraction θ_2' We close the set of equations by adding in the first equation (1),

$$k_0\sin\theta_0 = k_2'\sin\theta_2' \qquad \text{or} \qquad \boxed{\frac{\omega}{c}n_a\sin\theta_0 = k_2'\sin\theta_2'} \tag{5.5.31}$$

where we used $n_a = \dfrac{k_0 c}{\omega} = \sqrt{\mu_a\epsilon_a}$. Since the pair of equations (5.5.29) and (5.5.31) involve only the unknowns k_2' and θ_2', we can use them to eliminate k_2' and get a final single equation that determines θ_2' in terms of the angle of incidence θ_0.

Define the index of refraction of medium b as,

$$n_b \equiv \sqrt{\mu_b\epsilon_{b1}} \tag{5.5.32}$$

Then from Eqs. (5.5.29) and (5.5.31) we get,

$$\frac{\omega}{c}n_a\sin\theta_0 = \frac{\omega}{c}n_b\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\epsilon_{b2}^2}{\epsilon_{b1}^2\cos^2\theta_2'}}\,\right]^{1/2}\sin\theta_2' \tag{5.5.33}$$

or

$$\boxed{n_a\sin\theta_0 = n_b\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\epsilon_{b2}^2}{\epsilon_{b1}^2\cos^2\theta_2'}}\,\right]^{1/2}\sin\theta_2'} \tag{5.5.34}$$

This is the analog of Snell's Law for wave propagation into a non-transparent medium.

Consider two cases:

1) For a nearly transparent material, with $\epsilon_{b2} \ll \epsilon_{b1}$ we can expand in $\epsilon_{b2}/\epsilon_{b1}$ to get,

$$n_a \sin\theta_0 \approx n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{4\epsilon_{b1}^2 \cos^2\theta_2'}\right]^{1/2} \approx n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2 \cos^2\theta_2'}\right] \tag{5.5.35}$$

The second term in the square brackets is then a small correction to the familiar Snell's Law.

For $\epsilon_{b2} \ll \epsilon_{b1}$ we can solve the above iteratively.

To zeroth order: $n_a \sin\theta_0 = n_b \sin\theta_2' \quad \Rightarrow \quad \cos^2\theta_2' = 1 - \sin^2\theta_2' = 1 - \left(\dfrac{n_a}{n_b}\sin\theta_0\right)^2$.

Insert that in the left hand term of Eq. (5.5.35) to get the first order correction:

$$n_a \sin\theta_0 \approx n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2 \left(1 - \frac{n_a^2}{n_b^2}\sin^2\theta_0\right)}\right] \tag{5.5.36}$$

or

$$\sin\theta_2' = \frac{n_a}{n_b}\sin\theta_0 \left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2 \left(1 - \frac{n_a^2}{n_b^2}\sin^2\theta_0\right)}\right]^{-1} \approx \frac{n_a}{n_b}\sin\theta_0 \left[1 - \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2 \left(1 - \frac{n_a^2}{n_b^2}\sin^2\theta_0\right)}\right] \tag{5.5.37}$$

So

$$\sin\theta_2' < \frac{n_a}{n_b}\sin\theta_0 \tag{5.5.38}$$

The result is that θ_2' is *smaller* than the simple Snell's Law would predict.

2) For a good conductor, or absorbing region of a dielectric, where $\epsilon_{b2} \gg \epsilon_{b1}$, we can expand in $\epsilon_{b1}/\epsilon_{b2}$. To lowest order, Eq. (5.5.34) gives,

$$n_a \sin\theta_0 = \sqrt{\mu_b \epsilon_{b1}} \left[\frac{\epsilon_{b2}}{2\epsilon_{b1}\cos\theta_2'}\right]^{1/2} \sin\theta_2' = \sqrt{\frac{\mu_b \epsilon_{b2}}{2}} \frac{\sin\theta_2'}{\sqrt{\cos\theta_2'}} \tag{5.5.39}$$

This result is completely different from Snell's Law!

We could square the above to get,

$$n_a^2 \sin^2\theta_0 = \frac{\mu_b \epsilon_{b2}}{2} \frac{\sin^2\theta_2'}{\cos\theta_2'} = \frac{\mu_b \epsilon_{b2}}{2} \, \frac{1 - \cos^2\theta_2'}{\cos\theta_2'} \tag{5.5.40}$$

which we could rewrite as a quadratic equation to solve for $\cos\theta_2'$,

$$\cos^2\theta_2' + \left(\frac{2}{\mu_b \epsilon_{b2}}\right) \left(n_a^2 \sin^2\theta_0^2\right) \cos\theta_2' - 1 = 0 \tag{5.5.41}$$

The conclusion is that Snell's Law only holds if both media are transparent. If medium b is nearly transparent, Snell's Law will hold with a small correction that decreases the angle of transmission θ_2'. If medium b is highly absorbing, then the law for θ_2' bears no resemblance to Snell's Law.

Having found θ_2', one could then substitute into Eqs. (5.5.29) and (5.5.30) to determine k_2' and k_2''.

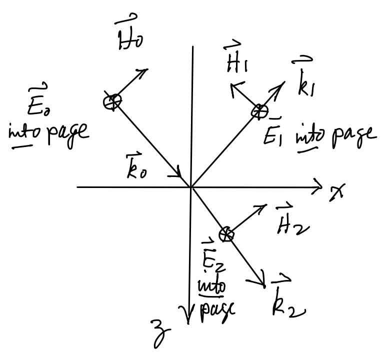
Reflection Coefficients

The results of the preceding sections all dealt with the issue of the frequency and wavevector of the reflected and transmitted waves. Here we will consider their amplitudes – how much of the incident wave is reflected, vs how much of it is transmitted into medium b. We will use the boundary conditions (b.c.) at the interface to determine this.

We will consider two cases: (1) where $\mathbf{E}_0$ is orthogonal to the plane of incidence, and (2) where $\mathbf{E}_0$ lies within the plane of incidence. Recall, the plane of incidence is the plane spanned by the wavevector $\mathbf{k}_0$ and the normal to the interface; in our calculation it is the xz plane.

We will do the calculation of these two cases in parallel. The results on the left will be case (1), the results on the right will be case (2)

(1) $\mathbf{E}_0 \perp$ plane of incidence



$\Rightarrow$ $\mathbf{H}_0$ is in plane of incidence

b.c.: tangential components of $\mathbf{E}$ are continuous

all $\mathbf{E}$'s are in $\hat{\mathbf{y}}$ direction; we can add them like scalars

$$\text{(i)} \quad E_0 + E_1 = E_2$$

b.c.: tangential components of $\mathbf{H}$ are continuous (since $\mathbf{j} = 0$)

$$(*) \quad H_{0x} + H_{1x} = H_{2x}$$

$$\text{Faraday:} \quad \frac{i\mu\omega}{c}\mathbf{H} = i\mathbf{k} \times \mathbf{E} \quad \Rightarrow \quad H_x = \frac{k_z c}{\omega\mu} E_y$$

substitute into $(*)$ to write H_x in terms of E

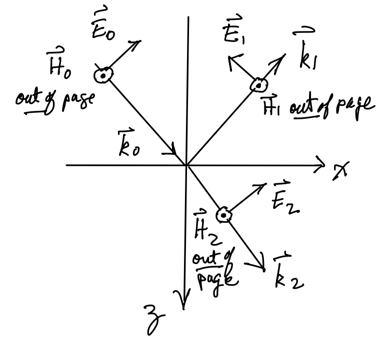
$$\text{(ii)} \qquad \Rightarrow \quad \frac{k_{0z}}{\mu_a}(E_0 - E_1) = \frac{k_{2a}}{\mu_b} E_2$$

solve (i) and (ii) for E_1 and E_2 in terms of E_0

$$E_1 = \frac{\mu_b\, k_{0z} - \mu_a\, k_{2z}}{\mu_b\, k_{0z} + \mu_a\, k_{2z}}\, E_0$$

$$E_2 = \frac{2\mu_b\, k_{0z}}{\mu_a\, k_{2z} + \mu_b\, k_{0z}}\, E_0$$

(2) $\mathbf{E}_0$ lies in the plane of incidence



$\Rightarrow$ $\mathbf{E}_0$ is in plane of incidence

b.c.: tangential components of $\mathbf{H}$ are continuous (since $\mathbf{j} = 0$)

all $\mathbf{H}$'s are in $\hat{\mathbf{y}}$ direction; we can add them like scalars

$$\text{(i)} \; H_0 + H_1 = H_2$$

b.c.: tangential components of $\mathbf{E}$ are continuous

$$(*) \quad E_{0x} + E_{1x} = E_{2x}$$

$$\text{Ampere:} \quad -\frac{\omega\epsilon}{c}\mathbf{E} = i\mathbf{k} \times \mathbf{H} \quad \Rightarrow \quad E_x = -\frac{k_z c}{\omega\epsilon} H_y$$

substitute into $(*)$ to write E_x in terms of H

$$\text{(ii)} \qquad \Rightarrow \quad \frac{k_{0z}}{\epsilon_a}(H_0 - H_1) = \frac{k_{2z}}{\epsilon_b} H_2$$

solve (i) and (ii) for H_1 and H_2 in terms of H_0

$$H_1 = \frac{\epsilon_b\, k_{0z} - \epsilon_a\, k_{2z}}{\epsilon_b\, k_{0z} + \epsilon_a\, k_{2z}}\, H_0$$

$$H_2 = \frac{2\epsilon_b\, k_{0z}}{\epsilon_b\, k_{2z} + \epsilon_b\, k_{0z}}\, H_0$$

We now define the reflection coefficient in terms of the transported energy,

$$R = \frac{|E_1|^2}{|E_0|^2} = \frac{|H_1|^2}{|H_0|^2} = \frac{\text{reflected energy current}}{\text{incident energy current}} \tag{5.5.42}$$

Then, for case (1) where $\mathbf{E}_0 \perp$ the plane of incidence, we have

$$R_\perp = \frac{|E_1|^2}{|E_0|^2} = \left| \frac{\mu_b\, k_{0z} - \mu_a\, k_{2z}}{\mu_b\, k_{0z} + \mu_a\, k_{2z}} \right|^2 \tag{5.5.43}$$

For case (2) where $\mathbf{E}_0 \parallel$ the plane of incidence, we have

$$R_\parallel = \frac{|H_1|^2}{|H_0|^2} = \left| \frac{\epsilon_b\, k_{0z} - \epsilon_a\, k_{2z}}{\epsilon_b\, k_{0z} + \epsilon_a\, k_{2z}} \right|^2 \tag{5.5.44}$$

Note, the above $R_\perp$ and $R_\parallel$ are correct for an *arbitrary* medium b (transparent, absorbing, reflecting).

Total Reflection

Consider now when medium b is in what we called the region of *total reflection* (this was denoted as region (3) in Notes 5-2).

In this region we had, from Notes 5-2, that $\mathrm{Im}\,[\epsilon_b] = \epsilon_{b2} \approx 0$, while $\mathrm{Re}\,[\epsilon_b] = \epsilon_{b1} < 0$. This led to $\mathbf{k}_2 = i\boldsymbol{\kappa}_2$, where $\boldsymbol{\kappa}_2$ is real valued, i.e., $\mathbf{k}_2$ is purely imaginary. Then,

$$R_\perp = \left| \frac{\mu_b\, k_{0z} - i\mu_a\, \kappa_{2z}}{\mu_b\, k_{0z} + i\mu_a\, \kappa_{2z}} \right|^2 \qquad \text{and} \qquad R_\parallel = \left| \frac{\epsilon_b\, k_{0z} - i\epsilon_a\, \kappa_{2z}}{\epsilon_b\, k_{0z} + i\epsilon_a\, \kappa_{2z}} \right|^2 \tag{5.5.45}$$

Both these are of the form $\left| \frac{a - ib}{a + ib} \right|^2$ where a and b are real valued. But in that case $\left| \frac{a - ib}{a + ib} \right|^2 = 1 \quad \Rightarrow \quad R_\perp = R_\parallel = 1.$

So this confirms that this region is indeed completely reflecting!

Medium b is Transparent

Next we consider the case when medium b is *transparent*. In this case ϵ_b is real and $\epsilon_b > 0$. Then we have,

$$k_{0z} = \frac{\omega}{c}\sqrt{\mu_a \epsilon_a}\cos\theta_0 = \frac{\omega}{c} n_a \cos\theta_0 \qquad \text{and} \qquad k_{2z} = \frac{\omega}{c}\sqrt{\mu_b \epsilon_b}\cos\theta_2 = \frac{\omega}{c} n_b \cos\theta_2 \tag{5.5.46}$$

Snell's Law holds and so $n_a \sin\theta_0 = n_b \sin\theta_2$. We can then write $R_\perp$ and $R_\parallel$ as functions of θ_0. For simplicity we will take $\mu_a = \mu_b = 1$. Then,

1) $\mathbf{E}_0 \perp$ plane of incidence,

$$R_\perp = \left(\frac{n_a \cos\theta_0 - n_b \cos\theta_2}{n_a \cos\theta_0 + n_b \cos\theta_2} \right)^2 \tag{5.5.47}$$

For normal incidence, $\theta_0 = 0$, then by Snell's Law we also have $\theta_2 = 0$, and so

$$R_\perp = \left(\frac{n_a - n_b}{n_a + n_b} \right)^2 \qquad \text{if } n_a = n_b \text{ then } R_\perp = 0\text{, so no reflection, as would be expected} \tag{5.5.48}$$

More generally, using Snell's Law to write $n_b = n_a \left(\frac{\sin\theta_0}{\sin\theta_2} \right)$, we have,

$$R_\perp = \left(\frac{\cos\theta_0 - \left(\frac{\sin\theta_0}{\sin\theta_2} \right)\cos\theta_2}{\cos\theta_0 + \left(\frac{\sin\theta_0}{\sin\theta_2} \right)\cos\theta_2} \right)^2 = \left(\frac{\sin\theta_2\cos\theta_0 - \sin\theta_0\cos\theta_2}{\sin\theta_2\cos\theta_0 + \sin\theta_0\sin\theta_2} \right)^2 = \left(\frac{\sin(\theta_0 - \theta_2)}{\sin(\theta_0 + \theta_2)} \right)^2 \tag{5.5.49}$$

2) $\mathbf{E}_0 \parallel$ plane of incidence,

$$R_\parallel = \left(\frac{\epsilon_b\, n_a\, \cos\theta_0 - \epsilon_a\, n_b\, \cos\theta_2}{\epsilon_b\, n_a\, \cos\theta_0 + \epsilon_a\, n_b\, \cos\theta_2} \right)^2 \tag{5.5.50}$$

Now use $n_b = \sqrt{\epsilon_b}$ when $\mu_b = 1$, and $n_a = \sqrt{\epsilon_a}$ when $\mu_a = 1$ to get,

$$R_\parallel = \left(\frac{n_b \cos\theta_0 - n_a \cos\theta_2}{n_b \cos\theta_0 + n_a \cos\theta_2}\right)^2 \tag{5.5.51}$$

For normal incidence $\theta_0 = \theta_2$ we again find,

$$R_\parallel = \left(\frac{n_b - n_a}{n_b + n_a}\right)^2 = R_\perp \tag{5.5.52}$$

This is as it must be, since when $\theta_0 = 0$, then there is no difference between the parallel and perpendicular cases since there is no plane of incidence as all $\mathbf{k}_0$, $\mathbf{k}_1$, and $\mathbf{k}_2$ are colinear.

More generally, using $n_b = n_a \left(\dfrac{\sin\theta_0}{\sin\theta_2}\right)^2$ from Snell's Law, we have,

$$R_\parallel = \left(\frac{\cos\theta_0 - \left(\dfrac{\sin\theta_2}{\sin\theta_0}\right)\cos\theta_2}{\cos\theta_0 + \left(\dfrac{\sin\theta_2}{\sin\theta_0}\right)\cos\theta_2}\right)^2 = \left(\frac{\sin\theta_0\cos\theta_0 - \sin\theta_2\cos\theta_2}{\sin\theta_0\cos\theta_0 + \sin\theta_2\cos\theta_2}\right)^2 = \left(\frac{\tan(\theta_0 - \theta_2)}{\tan(\theta_0 + \theta_2)}\right)^2 \tag{5.5.53}$$

The last step follows after quite a bit of algebra – I haven't found a simple way to show this!

Now if $\theta_0 + \theta_2 = \pi/2$, then $\tan(\theta_0 + \theta_2) \to \infty$, and $R_\parallel = 0$.

This occurs at an angle of incidence known as *Brewster's angle* $\theta_0 = \theta_B$. The condition determining θ_B is obtained from Snell's Law,

$$n_a \sin\theta_B = n_b \sin\theta_2 = n_b \sin(\pi/2 - \theta_B) = n_b \cos\theta_B \qquad \Rightarrow \qquad \boxed{\theta_B = \arctan\left(\frac{n_b}{n_a}\right)} \tag{5.5.54}$$

For an incident wave at θ_B, the reflected wave always has $\mathbf{E}_1$ perpendicular to the plane of incidence, since $R_\parallel = 0$. If the incoming wave has $\mathbf{E}_0 \parallel$ to the plane of incidence, then there is no reflected wave and the wave is completely transmitted. If $\mathbf{E}_0$ is in some general direction, then the reflected wave is always linearly polarized with $\mathbf{E}_1 \perp$ the plane of incidence. This is one method to create a polarized light wave.