Unit 6: Radiation and Fields from Moving Charges

In the previous unit we discussed the propagation of electromagnetic waves, which are solutions to Maxwell's equations in the *absence* of any source charges or currents, i.e., the macroscopic $\rho = 0$ and the macroscopic $\mathbf{j} = 0$. In this unit we add back sources to discuss what electromagnetic fields are produced by charges in motion, and how accelerated charges give rise to radiated electromagnetic waves. In this unit we leave the macroscopic Maxwell equations of unit 5, and the fields and sources will be taken as the microscopic quantities.

Unit 6-1: The Green's Function for the Wave Equation and the Liénard-Wiechert Potentials

We will work in the Lorenz gauge where the potentials satisfy,

$$\frac{1}{c}\frac{\partial \phi}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{A} = 0 \tag{6.1.1}$$

In this gauge, the inhomogeneous Maxwell equations become (see unit 1),

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho \qquad \text{and} \qquad \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j} \tag{6.1.2}$$

If we can solve the wave equation with a source (i.e., the inhomogeneous wave equation) then we in principle will have the solution to all electromagnetic problems where the source is a specified function of position and time. To do this we want to find the *Green's function* for the wave equation.

Recall from statics: $\nabla^2\phi = -4\pi\rho$

The electrostatic Green's function satisfies: $\nabla^2 G(\mathbf{r}) = -4\pi\delta(\mathbf{r})$

Then the solution for ϕ from a general source ρ is: $\phi(\mathbf{r}) = \int d^3r'\, G(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') + \phi_0(\mathbf{r})$

where $\phi_0(\mathbf{r})$ is any solution to the homogeneous equation, $\nabla^2\phi_0 = 0$.

For the case where the system fills all space and we want $\phi \to 0$ as $|\mathbf{r}| \to \infty$, the electrostatic Green's function is: $G(\mathbf{r}-\mathbf{r}') = \dfrac{1}{|\mathbf{r}-\mathbf{r}'|}$.

For the wave equation:

We want $G(\mathbf{r},t;\mathbf{r}',t')$ to satisfy the wave equation for a point source at position $\mathbf{r}'$ and time t',

$$\nabla^2 G(\mathbf{r},t;\mathbf{r}',t') - \frac{1}{c^2}\frac{\partial^2 G(\mathbf{r},t;\mathbf{r}',t')}{\partial^2 t} = -4\pi\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') \tag{6.1.3}$$

Once we have found the Green's function, the the solutions for general sources $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ will be given by,

$$\phi(\mathbf{r},t) = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} d^3r'\, G(\mathbf{r},t;\mathbf{r}',t')\,\rho(\mathbf{r}',t') + \phi_0(\mathbf{r},t) \tag{6.1.4}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} d^3r'\, G(\mathbf{r},t;\mathbf{r}',t')\,\mathbf{j}(\mathbf{r}',t') + \mathbf{A}_0(\mathbf{r},t) \tag{6.1.5}$$

where $\phi_0(\mathbf{r},t)$ and $\mathbf{A}_0(\mathbf{r},t)$ are any solutions of the *homogeneous* wave equation (for example they could describe an incoming wave).

For the case where the system fills all of space (as opposed to the case where the system is in a finite box), translational invariance gives,

$$G(\mathbf{r},t;\mathbf{r}',t') = G(\mathbf{r}-\mathbf{r}',t-t') \tag{6.1.6}$$

and the Green's function depends only on the distance from the observer at $(\mathbf{r}, t)$ and the source at $(\mathbf{r}', t')$.

If we express $G(\mathbf{r}, t)$ in terms of its Fourier transform $\tilde{G}(\mathbf{k}, \omega)$, it will be easy to solve for the Fourier amplitudes $\tilde{G}$. With

$$G(\mathbf{r}, t) = \int_{-\infty}^{\infty} \frac{d^3k d\omega}{(2\pi)^4} \tilde{G}(\mathbf{k}, \omega) \, \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{6.1.7}$$

then

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}, t) = \int_{-\infty}^{\infty} \frac{d^3k d\omega}{(2\pi)^4} \tilde{G}(\mathbf{k}, \omega) \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{6.1.8}$$

$$= \int_{-\infty}^{\infty} \frac{d^3k d\omega}{(2\pi)^4} \tilde{G}(\mathbf{k}, \omega) \left(-k^2 + \frac{\omega^2}{c^2}\right) \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{6.1.9}$$

$$= -4\pi\delta(\mathbf{r})\delta(t) = -4\pi \int_{-\infty}^{\infty} \frac{d^3k d\omega}{(2\pi)^4} \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{6.1.10}$$

Equating the Fourier amplitudes on either side of the equation we then get,

$$\left(-k^2 + \frac{\omega^2}{c^2}\right) \tilde{G}(\mathbf{k}, t) = -4\pi \qquad \Rightarrow \qquad \boxed{\tilde{G}(\mathbf{k}, t) = \frac{4\pi c^2}{c^2k^2 - \omega^2}} \tag{6.1.11}$$

Using Fourier transforms, we have converted the partial differential wave equation for G into an algebraic equation for $\tilde{G}$. This is why Fourier transforms are so useful for linear differential equations!

In the space and time domain, we then have,

$$G(\mathbf{r}, t) = \int_{-\infty}^{\infty} \frac{d^3k d\omega}{(2\pi)^4} \left(\frac{4\pi c^2}{c^2k^2 - \omega^2}\right) \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{6.1.12}$$

The integrand has poles at $\omega = \pm ck$.

To evaluate the above, we will use contour integration in the complex ω plane. To do that, we will have to know how to treat the poles, which lie on the real ω axis. We will treat the poles in such a way that the resulting $G(\mathbf{r}, t)$ has the desired *causal* behavior, i.e., $G(\mathbf{r}, t) = 0$ for $t < 0$, so that $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ depend only on the sources at *earlier* times $t' < t$.

We start by evaluating the $\mathbf{k}$ part of the integral in Eq. (6.1.12). Since $\tilde{G}(\mathbf{k}, \omega)$ depends only on $k = |\mathbf{k}|$, we write the integration over k-space in spherical coordinates, where θ is the angle between $\mathbf{k}$ and the observer at $\mathbf{r}$,

$$\int_{-\infty}^{\infty} d^3k \, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} \tilde{G}(\mathbf{k}, \omega) = 2\pi \int_0^{\pi} d\theta \, \sin\theta \int_0^{\infty} dk \, k^2 \, \mathrm{e}^{ikr\cos\theta} \tilde{G}(k, \omega) \qquad \text{use } \mu = -\cos\theta \text{ and } d\mu = d\theta \sin\theta \tag{6.1.13}$$

$$= 2\pi \int_{-1}^{1} d\mu \int_0^{\infty} dk \, k^2 \, \mathrm{e}^{ikr\mu} \tilde{G}(k, \omega) \qquad \text{do the integration over } \mu \tag{6.1.14}$$

$$= 4\pi \int_0^{\infty} dk \, k^2 \, \frac{\sin kr}{kr} \, \tilde{G}(k, \omega) \tag{6.1.15}$$

Now we do the integration over ω. Plugging in for $\tilde{G}(k, \omega)$ we get,

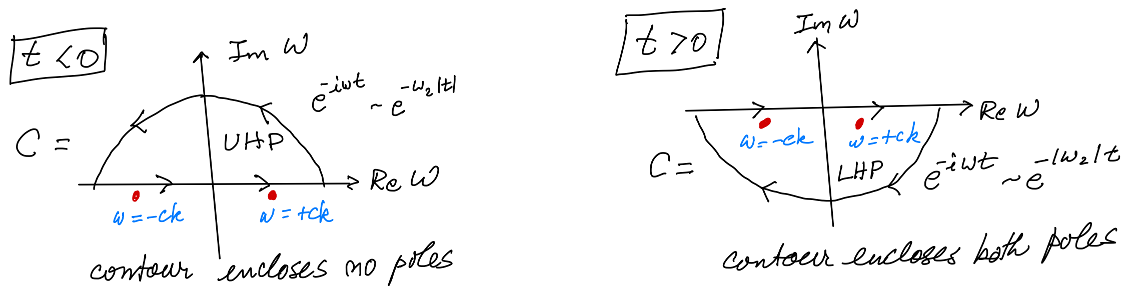
$$G(\mathbf{r}, t) = -\frac{c^2}{\pi^2} \int_0^{\infty} dk \, k^2 \, \frac{\sin kr}{kr} \oint_C d\omega \, \frac{\mathrm{e}^{-i\omega t}}{(\omega - ck)(\omega + ck)} \tag{6.1.16}$$

where the contour C will go along the real ω axis, and then close in the upper or lower half of the complex ω plane, depending on the sign of t as follows.

For complex $\omega = \omega_1 + i\omega_2$, the exponential factor in the integral $e^{-i\omega t} = e^{\omega_2 t} e^{i\omega_1 t}$. We want this to vanish as $|\omega_2| \to \infty$ on the infinite semi-circular part of C that closes the contour from the real ω axis, so that the integral over the contour C will be the same as the integral down the real ω axis. Therefore for $t > 0$, we must close the contour in the lower half of the complex ω plane, where $\omega_2 < 0$. But for $t < 0$, we must close the contour in the upper half of the complex ω plane, where $\omega_2 > 0$.

Now we need to decide how to treat the poles at $\omega = \pm ck$ on the real ω axis. From our discussion of the causal nature of the atomic polarizability $\alpha(\omega)$, we saw that the response will be causal when the poles lie in the lower half of the complex ω plane. We therefore regard the poles at $\omega = \pm ck$ as lying just *below* the real axis, so that they are in the lower half of the complex ω plane as desired for causality.

The contours C that we take for $t < 0$ and $t > 0$ are then as in the sketch below.



For $t < 0$, the contour C encloses no poles, so by the Cauchy residue theorem the integral vanishes, and we have $G(\mathbf{r}, t) = 0$ as desired for causality.

For $t > 0$, the contour C encloses both the poles, so we evaluate the integral using the Cauchy residue theorem. Looking at the ω part of the integral in Eq. (6.1.16), we have,

$$\text{For } t > 0: \qquad \oint_C d\omega \, \frac{e^{-i\omega t}}{(\omega - ck)(\omega + ck)} = -2\pi i \left[\frac{e^{-ickt}}{2ck} - \frac{e^{ickt}}{2ck} \right] = -2\pi \frac{\sin(ckt)}{ck} \tag{6.1.17}$$

The $(-)$ sign in front is because we go around the contour C in a clockwise direction. The first term in the square brackets is the residue at the pole $\omega = +ck$, while the second term is the residue at the pole $\omega = -ck$.

Using this result in Eq. (6.1.16) we then have,

$$G(\mathbf{r}, t) = \frac{2c}{\pi r} \int_0^\infty dk \, \sin(kr) \, \sin(ckt) = \frac{c}{\pi r} \int_{-\infty}^\infty dk \, \frac{\left(e^{ikr} - e^{-ikr}\right)\left(e^{ickt} - e^{-ickt}\right)}{(-4)} \tag{6.1.18}$$

$$= -\frac{c}{2r} \int_{-\infty}^\infty \frac{dk}{2\pi} \left[e^{i(r+ct)k} + e^{-i(r+ct)k} - e^{i(r-ct)k} - e^{-i(r-ct)k} \right] \tag{6.1.19}$$

$$= -\frac{c}{2r} \left[\delta(r + ct) + \delta(r + ct) - \delta(r - ct) - \delta(r - ct) \right] \tag{6.1.20}$$

In the first step we wrote sine in terms of complex exponentials, $\sin x = (e^{ix} - e^{-ix})/2i$, and used the fact that the integrand is symmetric in k to write $2\int_0^\infty dk = \int_{-\infty}^\infty dk$. In the second step we multiplied through the complex exponentials, and in the third step we identify the integrals as Dirac delta functions.

Finally, since by definition $r = |\mathbf{r}| \geq 0$, and we also are computing only for $t > 0$, then the argument of the delta function $\delta(r + ct)$ can never be zero! So this delta function always vanishes, and we have,

$$G(\mathbf{r}, t) = \frac{c}{r} \, \delta(r - ct) \tag{6.1.21}$$

Using the general result $\delta(ax) = \dfrac{\delta(x)}{a}$ we can then write,

$$G(\mathbf{r}, t) = \frac{c}{r}\,\delta(r - ct) = \frac{\delta(t - r/c)}{r} \tag{6.1.22}$$

and so the Green's function for the wave equation is,

$$\boxed{G(\mathbf{r}, t; \mathbf{r}', t') = \begin{cases} \dfrac{\delta\left(t - t' - \dfrac{|\mathbf{r} - \mathbf{r}'|}{c}\right)}{|\mathbf{r} - \mathbf{r}'|} & \text{for } t - t' > 0 \\ 0 & \text{for } t - t' < 0 \end{cases}} \tag{6.1.23}$$

We see that $G(\mathbf{r}, t; \mathbf{r}', t') \neq 0$ only on the *light cone* that emanates from the point $(\mathbf{r}', t')$, i.e., when $|\mathbf{r} - \mathbf{r}'| = c(t - t')$, so that a signal leaving point $\mathbf{r}'$ at time t', and traveling with the speed c, will reach the observer at position $\mathbf{r}$ at time t.

Using this Green's function, we now have the solution for the potentials from any specified sources $\rho(\mathbf{r}, t)$ and $\mathbf{j}(\mathbf{r}, t)$,

$$\phi(\mathbf{r}, t) = \phi_0(\mathbf{r}, t) + \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{t} dt'\, \frac{\delta\left(t - t' - |\mathbf{r} - \mathbf{r}'|/c\right)}{|\mathbf{r} - \mathbf{r}'|}\, \rho(\mathbf{r}', t') \tag{6.1.24}$$

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0(\mathbf{r}, t) + \frac{1}{c}\int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{t} dt'\, \frac{\delta\left(t - t' - |\mathbf{r} - \mathbf{r}'|/c\right)}{|\mathbf{r} - \mathbf{r}'|}\, \mathbf{j}(\mathbf{r}', t') \tag{6.1.25}$$

The Liénard-Wiechert Potentials

We can apply the above to the case where the source is a single point charge q moving on the trajectory $\mathbf{r}_0(t)$,

$$\rho(\mathbf{r}, t) = q\delta(\mathbf{r} - \mathbf{r}_0(t)) \qquad \text{and} \qquad \mathbf{j}(\mathbf{r}, t) = q\mathbf{v}(t)\delta(\mathbf{r} - \mathbf{r}_0(t)) \qquad \text{where } \mathbf{v} = \frac{d\mathbf{r}_0}{dt} \tag{6.1.26}$$

Then we can use the delta function in ρ to do the integrals over $\mathbf{r}'$ in Eq. (6.1.24) and get,

$$\phi(\mathbf{r}, t) = q\int_{-\infty}^{t} dt'\, \frac{\delta\left(t - t' - \dfrac{1}{c}|\mathbf{r} - \mathbf{r}_0(t')|\right)}{|\mathbf{r} - \mathbf{r}_0(t')|} \tag{6.1.27}$$

Because of the $\mathbf{r}_0(t')$ term in the argument of the delta function, the t' dependence of the argument is not of the simple form $t' - t_0$, that would allow us to do the integral over t' in a trivial way. Instead, if we write,

$$g(t') \equiv t' + \frac{1}{c}|\mathbf{r} - \mathbf{r}_0(t')| \tag{6.1.28}$$

then,

$$\phi(\mathbf{r}, t) = q\int_{-\infty}^{t} dt'\, \frac{\delta\left(t - g(t')\right)}{|\mathbf{r} - \mathbf{r}_0(t')|} \tag{6.1.29}$$

Now we transform the integration variable from t' to g,

$$\phi(\mathbf{r}, t) = q\int_{-\infty}^{g(t)} dg \left(\frac{dt'}{dg}\right) \frac{\delta\left(t - g(t')\right)}{|\mathbf{r} - \mathbf{r}_0(t')|} = \frac{q}{|\mathbf{r} - \mathbf{r}_0(t')|}\,\frac{1}{(dg/dt')}\Bigg|_{t' \text{ such that } g(t') = t} \tag{6.1.30}$$

Now

$$g(t') = t' + \frac{1}{c}\sqrt{[x - x_0(t')]^2 + [y - y_0(t')]^2 + [z - z_0(t')]^2} \tag{6.1.31}$$

so

$$\frac{dg}{dt'} = 1 + \frac{1}{c|\mathbf{r} - \mathbf{r}_0(t')|}\left[[x - x_0(t')]\left(-\frac{dx_0}{dt}\right) + [y - y_0(t')]\left(-\frac{dy_0}{dt}\right) + [z - z_0(t')]\left(-\frac{dz_0}{dt}\right)\right] \tag{6.1.32}$$

$$= 1 - \frac{1}{c}\hat{\mathbf{n}}(t') \cdot \mathbf{v}(t') \tag{6.1.33}$$

where

$$\hat{\mathbf{n}}(t') \equiv \frac{\mathbf{r} - \mathbf{r}_0(t')}{|\mathbf{r} - \mathbf{r}_0(t')|} \qquad \text{is the unit vector pointing from } \mathbf{r}_0(t') \text{ to } \mathbf{r} \tag{6.1.34}$$

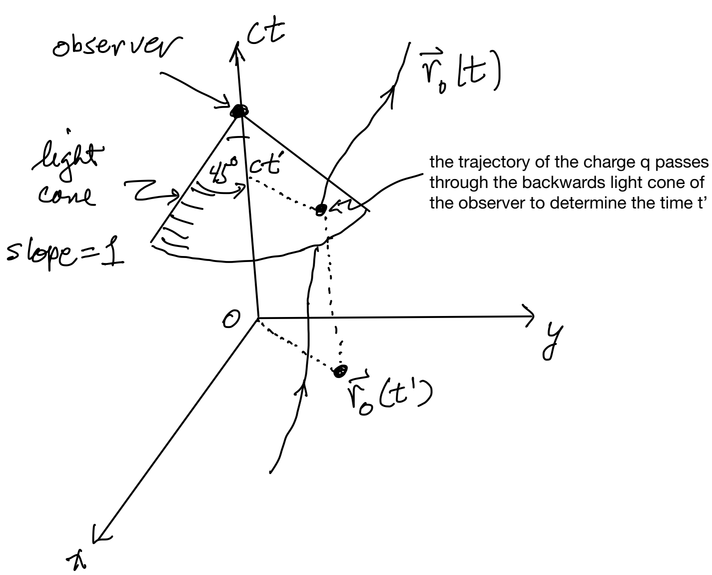
Thus we have for $\phi(\mathbf{r}, t)$, and by a similar calculation we have for $\mathbf{A}(\mathbf{r}, t)$, the *Liénard-Wiechert Potentials*,

$$\phi(\mathbf{r}, t) = \frac{q}{|\mathbf{r} - \mathbf{r}_0(t')|\,[1 - \hat{\mathbf{n}}(t') \cdot \mathbf{v}(t')/\,c\,]} \qquad \text{and} \qquad \mathbf{A}(\mathbf{r}, t) = \frac{q\mathbf{v}(t')/c}{|\mathbf{r} - \mathbf{r}_0(t')|\,[1 - \hat{\mathbf{n}}(t') \cdot \mathbf{v}(t')/c\,]} \tag{6.1.35}$$

where t', called the *retarded time*, is determined by the condition

$$t - t' = \frac{1}{c}|\mathbf{r} - \mathbf{r}_0(t')| \tag{6.1.36}$$

This is illustrated graphically in the space-time diagram shown below



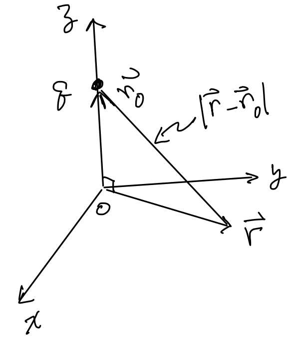
For an observer at $\mathbf{r} = 0$ at time t, the time t' is determined from the point where the charge's trajectory $\mathbf{r}_0(t)$ passes through the observer's backward directed light cone. The velocity and position of the charge at time t' determines the potentials seen by the observer.

A Charge Moving at Constant Velocity

In general, for an arbitrary charge trajectory $\mathbf{r}_0(t)$, the Liénard-Wiechert Potentials can be difficult to calculate, because of the difficulty in determining the time t'. However we can do the calculation for the simple case of a charge moving at constant velocity. We take the charge to be moving along the $\hat{\mathbf{z}}$ axis with the trajectory,

$$\mathbf{r}_0(t) = vt\,\hat{\mathbf{z}} \qquad \text{with} \qquad \mathbf{v} = \frac{d\mathbf{r}_0}{dt} = v\,\hat{\mathbf{z}} \tag{6.1.37}$$

For an observer at position $\mathbf{r}$ in the xy plane at $z = 0$, the fields at time t will be determined by the charge at the earlier time t' such that,



$$t - t' - \frac{|\mathbf{r} - \mathbf{r}_0(t')|}{c} = 0 \qquad \Rightarrow \qquad t - t' - \frac{\sqrt{r^2 + v^2 t'^2}}{c} = 0 \tag{6.1.38}$$

$$\Rightarrow \qquad (t - t')^2 = t^2 + t'^2 - 2tt' = \frac{r^2 + v^2 t'^2}{c^2} \tag{6.1.39}$$

$$\Rightarrow \qquad (1 - v^2/c^2)t'^2 - 2tt' + t^2 - r^2/c^2 = 0 \tag{6.1.40}$$

Let

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \tag{6.1.41}$$

Then the above becomes the quadratic equation in t',

$$t'^2 - 2\gamma^2 t t' + \frac{\gamma^2}{c^2}(c^2 t^2 - r^2) = 0 \tag{6.1.42}$$

We can now solve for t' using the quadratic formula,

$$t' \;=\; \gamma^2 t \;\pm\; \sqrt{\gamma^4 t^2 - \gamma^2 t^2 + \gamma^2 r^2/c^2} \;=\; \gamma^2 t \;\pm\; \gamma\sqrt{[\gamma^2 - 1]t^2 + r^2/c^2} \tag{6.1.43}$$

Now use $\gamma^2 - 1 = \dfrac{1}{1 - v^2/c^2} - 1 = \dfrac{v^2/c^2}{1 - v^2/c^2} = \gamma^2 \dfrac{v^2}{c^2}$, and we have

$$t' \;=\; \gamma^2 t \;\pm\; \gamma\sqrt{\gamma^2 \frac{v^2}{c^2} t^2 + \frac{r^2}{c^2}} \;=\; \gamma^2 t \;\pm\; \frac{\gamma^2}{c}\sqrt{v^2 t^2 + \frac{r^2}{\gamma^2}} \tag{6.1.44}$$

To choose which $(\pm)$ sign above is correct, consider $t = 0$. The solution should then give $t' < 0$, since t' is always earlier than t by causality. Only the $(-)$ sign gives this, so that is the correct solution.

$$t' \;=\; \gamma^2 t \;-\; \frac{\gamma^2}{c}\sqrt{v^t t^2 + \frac{r^2}{\gamma^2}} \tag{6.1.45}$$

Now we have for the scalar potential,

$$\phi(\mathbf{r}, t) = \frac{q}{|\mathbf{r} - \mathbf{r}_0(t')|\,[1 - \hat{\mathbf{n}}(t') \cdot \mathbf{v}/c]} \qquad \text{where} \qquad \hat{\mathbf{n}}(t') \equiv \frac{\mathbf{r} - \mathbf{r}_0(t')}{|\mathbf{r} - \mathbf{r}_0(t')|} \tag{6.1.46}$$

Here

$$|\mathbf{r} - \mathbf{r}_0(t')| \;=\; \sqrt{r^2 + v^2 t'^2} \;=\; c(t - t') \qquad \text{from the condition that determines } t' \tag{6.1.47}$$

$$(\mathbf{r} - \mathbf{r}_0(t')) \cdot \mathbf{v} \;=\; -\mathbf{r}_0(t') \cdot \mathbf{v} \qquad \text{since } \mathbf{r} \text{ is in the } xy \text{ plane and } \mathbf{v} = v\hat{\mathbf{z}}, \text{ so } \mathbf{r} \cdot \mathbf{v} = 0 \tag{6.1.48}$$

$$= \; -v^2 t' \qquad \text{since } \mathbf{r}_0(t') = \mathbf{v}t' \tag{6.1.49}$$

Putting the pieces together we get,

$$\phi(\mathbf{r},t) \;=\; \frac{q}{c(t-t')\left[1+\dfrac{v^2t'}{c^2(t-t')}\right]} \;=\; \frac{q}{c(t-t')+\dfrac{v^2t'}{c}} \;=\; \frac{q}{c\left[t-\left(1-\dfrac{v^2}{c^2}\right)t'\right]} \qquad (6.1.50)$$

$$=\; \frac{q}{c\left(t-\dfrac{t'}{\gamma^2}\right)} \;=\; \frac{q}{c\,\dfrac{1}{c}\sqrt{v^2t^2+\dfrac{r^2}{\gamma^2}}} \qquad \text{where we used Eq. (6.1.45)} \qquad (6.1.51)$$

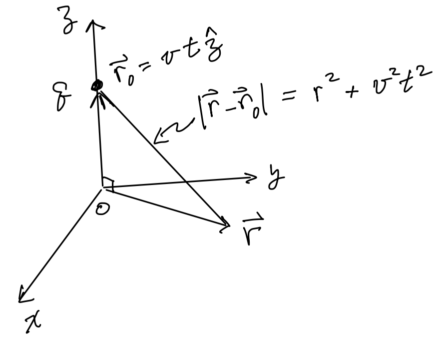
$$\phi(\mathbf{r},t) \;=\; \frac{q}{\sqrt{v^2t^2+r^2/\gamma^2}} \qquad (6.1.52)$$

And by a similar calculation we get,

$$\mathbf{A}(\mathbf{r},t) \;=\; \frac{q\mathbf{v}}{c\sqrt{v^2t^2+r^2/\gamma^2}} \;=\; \frac{\mathbf{v}}{c}\phi(\mathbf{r},t) \qquad (6.1.53)$$

Note, the scalar potential ϕ at the observer at position $\mathbf{r}$ in the xy plane at time t, when the charge q is at position $\mathbf{r}_0(t)=vt\,\hat{\mathbf{z}}$ on the $\hat{\mathbf{z}}$ axis, looks almost like the *static* Coulomb potential, which would be,

$$\frac{q}{|\mathbf{r}-\mathbf{r}_0(t)|} \;=\; \frac{q}{\sqrt{v^2t^2+r^2}} \qquad (6.1.54)$$



Instead, it is

$$\frac{q}{\sqrt{v^2t^2+r^2/\gamma^2}} \qquad (6.1.55)$$

This looks like the directions transverse to the direction of the charge's motion have contracted by a factor γ.

Such considerations led Lorentz to discover the Lorentz transformation *before* Einstein proposed his theory of relativity!

Note, the above Eqs. (6.1.52) and (6.1.53) dealt with the particular case of an observer in the xy plane at $z=0$. But we can get the general result for an observer at any (x,y,z) by noting that what the observer at height z sees at time t will be the same as an observer at height $z=0$ saw at the earlier time $t-z/v$. We thus have,

$$\phi(x,y,z,t)=\phi(x,y,0,t-z/v) \qquad \mathbf{A}(x,y,z,t)=\mathbf{A}(x,y,0,t-z/v) \qquad (6.1.56)$$

After a little bit of algebra, you can show that this general result can be written as,

$$\phi(\mathbf{r},t)=\frac{q}{\sqrt{(\mathbf{r}-\mathbf{v}t)^2+(\mathbf{r}\cdot\mathbf{v}/c)^2-(rv/c)^2}} \qquad \text{and} \qquad \mathbf{A}(\mathbf{r},t)=\frac{\mathbf{v}}{c}\phi(\mathbf{r},t) \qquad (6.1.57)$$

Having expressed the potentials for any observer position $\mathbf{r}=(x,y,z)$, we can now take the necessary derivatives to compute the corresponding electric and magnetic fields. After some more algebra we get,

$$\mathbf{E}=-\boldsymbol{\nabla}\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}=\frac{q\,(\mathbf{r}-\mathbf{v}t)\,\gamma^{-2}}{\left[(\mathbf{r}-\mathbf{v}t)^2+(\mathbf{r}\cdot\mathbf{v}/c)^2-(rv/c)^2\right]^{3/2}} \;\rightarrow\; \frac{q\,(\mathbf{r}-\mathbf{v}t)}{|\mathbf{r}-\mathbf{v}t|^3} \quad \text{as } v/c\rightarrow 0 \qquad (6.1.58)$$

and

$$\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}=\boldsymbol{\nabla}\times\left[\frac{\mathbf{v}}{c}\phi\right]=-\frac{\mathbf{v}}{c}\times\boldsymbol{\nabla}\phi=\frac{\mathbf{v}}{c}\times\left[\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right]=\frac{\mathbf{v}}{c}\times\mathbf{E} \qquad (6.1.59)$$

where the last step follows since $\dfrac{\partial \mathbf{A}}{\partial t} \propto \mathbf{v}$ and so $\mathbf{v} \times \dfrac{\partial \mathbf{A}}{\partial t} = 0$. We thus have,

$$\mathbf{B} \;\to\; q\left(\frac{\mathbf{v}}{c}\right) \times \frac{(\mathbf{r} - \mathbf{v}t)}{|\mathbf{r} - \mathbf{v}t|^3} \qquad \text{as } v/c \to 0 \tag{6.1.60}$$

So as $v/c \to 0$, the electric field looks just like the static Coulomb field, while the magnetic field looks like the Biot-Savart law with $\mathbf{j} = q\mathbf{v}\,\delta(\mathbf{r} - \mathbf{v}t)$.