Unit 6-2: Radiation by a Localized Oscillating Source

In the previous section we saw that the fields produced by a charge q moving with *constant* velocity $\mathbf{v}$ looked qualitatively like static fields; the decay roughly as $\sim 1/r^2$ with r the distance from the source. There are no electromagnetic waves produced. This is as it must be, since by the theory of special relativity we can always transform to an inertial frame of reference in which the charge q is at rest, and therefore in that frame of reference the fields are static. In this section we want to see how to produce electromagnetic waves. We will see that an oscillating charge distribution does the trick.

Consider a localized current source $\mathbf{j}(\mathbf{r},t)$. From the previous section we know that the resulting vector potential is,

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{t} dt' \, \frac{\delta(t-t'-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}\, \mathbf{j}(\mathbf{r}',t') \tag{6.2.1}$$

We will consider a current that has a pure harmonic oscillation with frequency ω,

$$\mathbf{j}(\mathbf{r},t) = \mathrm{Re}\left[\,\mathbf{j}_\omega(\mathbf{r})\mathrm{e}^{-i\omega t}\,\right] \tag{6.2.2}$$

The vector potential will then oscillate with the same frequency,

$$\mathbf{A}(\mathbf{r},t) = \mathrm{Re}\left[\,\mathbf{A}_\omega(\mathbf{r})\mathrm{e}^{-i\omega t}\,\right] \tag{6.2.3}$$

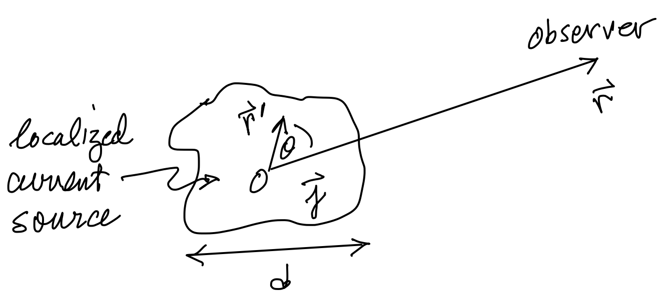
Substituting into Eq. (6.2.1), and using the delta function to do the integration over t', we then get,

$$\mathbf{A}_\omega(\mathbf{r})\mathrm{e}^{-i\omega t} = \frac{1}{c}\int_{-\infty}^{\infty} d^3r'\, \mathbf{j}_\omega(\mathbf{r}')\mathrm{e}^{-i\omega t}\, \frac{\mathrm{e}^{i\omega|\mathbf{r}-\mathbf{r}'|/c}}{|\mathbf{r}-\mathbf{r}'|} \tag{6.2.4}$$

$$\mathbf{A}_\omega(\mathbf{r}) = \frac{1}{c}\int_{-\infty}^{\infty} d^3r'\, \mathbf{j}_\omega(\mathbf{r}')\, \frac{\mathrm{e}^{i\omega|\mathbf{r}-\mathbf{r}'|/c}}{|\mathbf{r}-\mathbf{r}'|} \tag{6.2.5}$$

Next we will assume that the current source is localized, i.e., $\mathbf{j}_\omega(\mathbf{r}) \approx 0$ for $|\mathbf{r}| > d$, where d is the length that characterizes the extent of the current source. For an observer that is far away from the source, we can then expand the denominator in r'/r.

Approximation (1): Far from the source, $r \gg d$



we have,

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta} \tag{6.2.6}$$

$$= r\sqrt{1 + \left(\frac{r'}{r}\right)^2 - \frac{2r'}{r}\cos\theta} \tag{6.2.7}$$

$$= r\left(1 - \frac{r'}{r}\cos\theta\right) + O\left(\frac{r'}{r}\right)^2 \tag{6.2.8}$$

$$\approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' \tag{6.2.9}$$

where we expanded the square root only to lowest order in r'/r using $\sqrt{1+\delta} \approx 1 + \delta/2$.

With this, the vector potential becomes,

$$\mathbf{A}_\omega(\mathbf{r}) = \frac{1}{c}\int_{-\infty}^{\infty} d^3r'\, \mathbf{j}_\omega(\mathbf{r}')\, \frac{\mathrm{e}^{ik(r-\hat{\mathbf{r}}\cdot\mathbf{r}')}}{r - \hat{\mathbf{r}}\cdot\mathbf{r}'} \qquad \text{where we define } k \equiv \omega/c \tag{6.2.10}$$

$$= \frac{\mathrm{e}^{ikr}}{cr}\int_{-\infty}^{\infty} d^3r'\, \mathbf{j}_\omega(\mathbf{r}')\, \frac{\mathrm{e}^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}}{1 - \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r}} \approx \frac{\mathrm{e}^{ikr}}{cr}\int_{-\infty}^{\infty} d^3r'\, \mathbf{j}_\omega(\mathbf{r}')\, \mathrm{e}^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\left(1 + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r}\right) \tag{6.2.11}$$

where in the last step we expanded the denominator to lowest order, $1/(1-\delta) \approx 1+\delta$.

The expression above already shows a very important result. When we put back the time dependence, the vector potential has the form,

$$\mathbf{A}(\mathbf{r},t) = \mathrm{Re}\left[\,\mathbf{A}_\omega(\mathbf{r})\mathrm{e}^{-i\omega t}\,\right] = \mathrm{Re}\left[\frac{\mathrm{e}^{i(kr-\omega t)}}{r}\mathbf{f}(\hat{\mathbf{r}})\right] \tag{6.2.12}$$

where $\mathbf{f}(\hat{\mathbf{r}})$ represent the integral in Eq. (6.2.11), and depends only on the orientation of the observer $\hat{\mathbf{r}}$. The important factor, however, is,

$$\frac{\mathrm{e}^{i(kr-\omega t)}}{r} \tag{6.2.13}$$

This represents a *spherical* wave, traveling outward from the source in the radial direction with speed $\omega/k = c$. The factor $1/r$ in this term is important for energy conservation. Since the energy carried by a wave goes as the amplitude squared, the energy flowing through the surface of a sphere of radius r centered about the origin will be proportional to the surface area of the sphere $\sim r^2$ times the square of the amplitude. The $1/r$ factor in the amplitude thus guarantees that the energy flux through such a surface is $\sim (r^2)(1/r^2)$ will be independent of r. So the energy flowing through the surface of a sphere of radius r will be the same as the energy flowing through a sphere of radius $2r$, etc. No energy gets lost in between!

We thus see that an oscillating current source results in a radially outward traveling spherical electromagnetic wave! The factor $\mathbf{f}(\hat{\mathbf{r}})$ gives how the amplitude of this wave varies as one looks in different directions $\hat{\mathbf{r}}$.

Approximation (2): Long wavelength, $\boldsymbol{\lambda \gg d}$

Now we assume that the wavelength of the emitted spherical wave, $\lambda = 2\pi/k$, is large compared to the spatial extent of the source, $\lambda \gg d \;\Rightarrow\; kd \ll 1 \;\Rightarrow\; (\omega/c)d \ll 1 \;\Rightarrow\; (d/\tau) \ll c$, where $\tau = 2\pi/\omega$ is the period of oscillation.

Since the maximum displacement a charge may have during one period of oscillation is just the length of the distribution d, then d/τ is the maximum speed of the oscillating charges. So $\lambda \gg d \;\Rightarrow\; (d/\tau) \ll c$ is equivalent to a *non-relativistic* approximation. The charges are all moving slowly compared to the speed of light.

For $kd \ll 1$ we can write $\mathrm{e}^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \approx 1 - ik\hat{\mathbf{r}}\cdot\mathbf{r}' + O(kd)^2$. We then have,

$$\mathbf{f}(\hat{\mathbf{r}}) \equiv \frac{1}{c}\int_{-\infty}^{\infty} d^3r'\,\mathbf{j}_\omega(\mathbf{r}')\,\mathrm{e}^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\left(1+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r}\right) \approx \frac{1}{c}\int_{-\infty}^{\infty} d^3r'\,\mathbf{j}_\omega(\mathbf{r}')\,\left(1-ik\hat{\mathbf{r}}\cdot\mathbf{r}'\right)\left(1+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r}\right) \tag{6.2.14}$$

$$\approx \frac{1}{c}\int_{-\infty}^{\infty} d^3r'\,\mathbf{j}_\omega(\mathbf{r}')\,\left[1+\hat{\mathbf{r}}\cdot\mathbf{r}'\left(\frac{1}{r}-ik\right)\right] + \text{higher order terms in } d/r \text{ or } kd \tag{6.2.15}$$

We will write this as,

$$\mathbf{f}(\hat{\mathbf{r}}) = \mathbf{I}_1 + \left(\frac{1}{r}-ik\right)\mathbf{I}_2 \tag{6.2.16}$$

where

$$\mathbf{I}_1 \equiv \frac{1}{c}\int_{-\infty}^{\infty} d^3r'\,\mathbf{j}_\omega(\mathbf{r}') \qquad \text{and} \qquad \mathbf{I}_2 \equiv \frac{1}{c}\int_{-\infty}^{\infty} d^3r'\,(\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathbf{j}_\omega(\mathbf{r}') \tag{6.2.17}$$

Consider first $\mathbf{I}_1$ Recall, $\mathbf{I}_1$ vanishes in *magnetostatics*.

The ith component of the vector integral $\mathbf{I}_1$ is given by

$$cI_{1i} = \int d^3r'\,j_{\omega i}(\mathbf{r}') \tag{6.2.18}$$

To rewrite $j_{\omega i}(\mathbf{r}')$ we will integrate by parts, as we have done before in Notes 2-5 when discussing the magnetic dipole approximation in statics. Since $\boldsymbol{\nabla}' r_i' = \hat{\mathbf{e}}_i$, with $\hat{\mathbf{e}}_i$ the unit vector in direction i, then,

$$j_{\omega i} = (\boldsymbol{\nabla}' r_i')\cdot\mathbf{j}_\omega = \boldsymbol{\nabla}'\cdot(r_i'\mathbf{j}_\omega) - r_i'\boldsymbol{\nabla}'\cdot\mathbf{j}_\omega = \boldsymbol{\nabla}'\cdot(r_i'\mathbf{j}_\omega) - r_i'(i\omega\rho_\omega) \quad \text{since by charge conservation } \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0 \tag{6.2.19}$$

So

$$\int_V d^3r'\, j_{\omega i} = \int_V d^3r'\, \boldsymbol{\nabla}'\cdot(r_i'\mathbf{j}_\omega) - i\omega\int_V d^3r'\, r_i'\,\rho_\omega = \oint_S da'\,\hat{\mathbf{n}}\cdot(r_i'\mathbf{j}_\omega) - i\omega\int_V d^3r'\, r_i'\,\rho_\omega \tag{6.2.20}$$

As we take the volume $V\to\infty$, then the bounding surface $S\to 0$, and since $\mathbf{j}$ is localized it vanishes on S and so the surface integral vanishes. Here we also used $\rho(\mathbf{r},t) = \rho_\omega(\mathbf{r})\mathrm{e}^{-i\omega t}$, so that $\partial\rho/\partial t \to -i\omega\rho_\omega$. Thus,

$$\int_{-\infty}^{\infty} d^3r'\,\mathbf{j}_\omega(\mathbf{r}') = -i\omega\int_{-\infty}^{\infty} d^3r'\,\mathbf{r}'\,\rho_\omega(\mathbf{r}') = -i\omega\,\mathbf{p}_\omega \tag{6.2.21}$$

where $\mathbf{p}_\omega$ is the amplitude of the oscillating electric dipole moment, $\mathbf{p}(t) = \mathbf{p}_\omega\mathrm{e}^{-i\omega t}$. So finally,

$$\mathbf{I}_1 = -\frac{i\omega}{c}\mathbf{p}_\omega \tag{6.2.22}$$

This term $\mathbf{I}_1$ gives the *electric dipole approximation* $\mathbf{A}_{E1}$ for the vector potential $\mathbf{A}$. The amplitude of the oscillation of $\mathbf{A}$ in the electric dipole approximation is,

$$\boxed{\mathbf{A}_{E1}(\mathbf{r}) = \frac{\mathrm{e}^{ikr}}{r}\left(-\frac{i\omega}{c}\mathbf{p}_\omega\right) = \frac{\mathrm{e}^{ikr}}{r}(-i\,k\,\mathbf{p}_\omega)} \qquad \text{where } \omega = kc\text{, and the subscript "}E1\text{" refers to } \textit{electric dipole} \tag{6.2.23}$$

<u>Consider next $\mathbf{I}_2$</u>

$$\mathbf{I}_2 = \frac{1}{c}\int d^3r'\,(\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathbf{j}_\omega(\mathbf{r}') = \frac{1}{c}\hat{\mathbf{r}}\cdot\int d^3r'\,[\mathbf{r}'\,\mathbf{j}_\omega(\mathbf{r}')] \tag{6.2.24}$$

The last term above is a *tensor*. We saw this tensor when we did the magnetic dipole approximation in Notes 2-5, and also when we derived the macroscopic Maxwell equations in Notes 3-2. There we found,

$$\int d^3r'\,[\mathbf{r}'\,\mathbf{j}_\omega(\mathbf{r}')] = -\int d^3r'\,[\mathbf{j}_\omega(\mathbf{r}')\,\mathbf{r}'] - \int d^3r'\,\mathbf{r}'\,\mathbf{r}'\,[\boldsymbol{\nabla}'\cdot\mathbf{j}_\omega(\mathbf{r}')] \tag{6.2.25}$$

$$= \frac{1}{2}\int d^3r'\,[\mathbf{r}'\,\mathbf{j}_\omega \;-\; \mathbf{j}_\omega\,\mathbf{r}'] - \frac{1}{2}\int d^3r'\,i\omega\,\mathbf{r}'\,\mathbf{r}'\,\rho_\omega \tag{6.2.26}$$

So

$$\mathbf{I}_2 = \frac{1}{2c}\int d^3r'\Big[(\hat{\mathbf{r}}\cdot\mathbf{r}')\mathbf{j}_\omega - (\hat{\mathbf{r}}\cdot\mathbf{j}_\omega)\mathbf{r}'\Big] - \frac{i\omega\hat{\mathbf{r}}}{2c}\cdot\int d^3r'\,[\mathbf{r}'\,\mathbf{r}']\rho_\omega \tag{6.2.27}$$

$$= -\frac{1}{2c}\int d^3r'\,[\hat{\mathbf{r}}\times(\mathbf{r}'\times\mathbf{j}_\omega)] - \frac{i\omega\hat{\mathbf{r}}}{2c}\cdot\int d^3r'[\mathbf{r}'\,\mathbf{r}']\rho_\omega \tag{6.2.28}$$

$$= -\hat{\mathbf{r}}\times\mathbf{m}_\omega - \frac{i\omega}{6c}\,\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}'_\omega \tag{6.2.29}$$

where

$$\mathbf{m}_\omega = \frac{1}{2c}\int d^3r'\,\mathbf{r}'\times\mathbf{j}_\omega(\mathbf{r}') \tag{6.2.30}$$

is the amplitude of the oscillating magnetic dipole moment, $\mathbf{m}(t) = \mathbf{m}_\omega \mathrm{e}^{-i\omega t}$, and

$$\overleftrightarrow{\mathbf{Q}}'_\omega = \int d^3r'\, 3[\mathbf{r}'\,\mathbf{r}']\,\rho_\omega(\mathbf{r}') \tag{6.2.31}$$

is the amplitude of an oscillating second moment of the charge distribution that is similar to the electric quadrupole moment. We saw $\overleftrightarrow{\mathbf{Q}}'$ already in Notes 3-1. $\overleftrightarrow{\mathbf{Q}}'$ is similar to $\overleftrightarrow{\mathbf{Q}}$, except the latter is traceless while the former is not.

The traceless electric quadrupole moment that we saw in the electric multipole expansion of Notes 2-4 was,

$$\overleftrightarrow{\mathbf{Q}}_\omega = \int d^3r' \left(3\mathbf{r}'\,\mathbf{r}' - \mathbf{r}'^2\overleftrightarrow{\mathbf{I}}\right)\rho_\omega(\mathbf{r}') \qquad \text{where } \overleftrightarrow{\mathbf{I}} \text{ is the identity tensor} \tag{6.2.32}$$

So we can write $\overleftrightarrow{\mathbf{Q}}'_\omega$ in terms of $\overleftrightarrow{\mathbf{Q}}_\omega$,

$$\overleftrightarrow{\mathbf{Q}}'_\omega = \overleftrightarrow{\mathbf{Q}}_\omega + \overleftrightarrow{\mathbf{I}} \int d^3r'\,\mathbf{r}'^2\rho_\omega(\mathbf{r}') = \overleftrightarrow{\mathbf{Q}}_\omega + \overleftrightarrow{\mathbf{I}}C_\omega \tag{6.2.33}$$

where $C_\omega = \int d^3r'\,\mathbf{r}'^2\rho_\omega$ is a scalar.

So finally we have,

$$\mathbf{I}_2 = -\hat{\mathbf{r}} \times \mathbf{m}_\omega - \frac{i\omega}{6c}\,\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}_\omega - \frac{i\omega}{6c}\hat{\mathbf{r}}\,C_\omega \tag{6.2.34}$$

We will soon see that the term involving C_ω does not contribute to the fields $\mathbf{E}$ and $\mathbf{B}$.

The term then $\mathbf{I}_2$ gives the *magnetic dipole* and *electric quadrupole* contributions to the vector potential $\mathbf{A}_\omega(\mathbf{r})$.

$$\boxed{\mathbf{A}_{M1}(\mathbf{r}) = \frac{\mathrm{e}^{ikr}}{r}\left(\frac{1}{r} - ik\right)(-\hat{\mathbf{r}}\times\mathbf{m}_\omega)} \qquad \text{where the subscript “}M1\text{” refers to the } \textit{magnetic dipole} \tag{6.2.35}$$

and

$$\boxed{\mathbf{A}_{E2}(\mathbf{r}) = \frac{\mathrm{e}^{ikr}}{r}\left(\frac{1}{r} - ik\right)\left(-\frac{i\omega}{6c}\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}_\omega\right)} \qquad \text{where the subscript “}E2\text{” refers to the } \textit{electric quadrupole} \tag{6.2.36}$$

The last term in $\mathbf{I}_2$,

$$\frac{\mathrm{e}^{ikr}}{r}\left(\frac{1}{r} - ik\right)\left(-\frac{i\omega}{6c}\hat{\mathbf{r}}\,C_\omega\right) \tag{6.2.37}$$

can be ignored. It is a radial function $\sim \hat{\mathbf{r}}$ and so its curl always vanishes, therefore it gives no contribution to $\mathbf{B}$. Far from the source where $\mathbf{j} = 0$, Ampere's law is $-\frac{i\omega}{c}\mathbf{E}_\omega = i\mathbf{k}\times\mathbf{B}_\omega$. Since $\mathbf{E}$ is thus determined from $\mathbf{B}$, the term involving C_ω will also give no contribution to $\mathbf{E}$. This term can be formally removed by a gauge transformation. We will just choose to ignore it since it has no effect on the fields $\mathbf{B}$ and $\mathbf{E}$.

So with the above approximations: (1) $r \gg d$, observer is far from the source, and (2) $\lambda \gg d$, long wavelength, non-relativistic, we have,

$$\mathbf{A}_\omega(\mathbf{r}) = \mathbf{A}_{E1}(\mathbf{r}) + \mathbf{A}_{M1}(\mathbf{r}) + \mathbf{A}_{E2}(\mathbf{r}) \tag{6.2.38}$$

Keeping higher order terms would give the magnetic quadrupole, electric octupole, etc., contributions.

Approximation (3): Radiation Zone, $r \gg \lambda$

In our last approximation we assume that the observer is far from the source not only on the length scale d of the source itself, but on the length scale λ of the emitted radiation. Recall, by Approximation (2) we have $\lambda \gg d$, so now we have, $r \gg \lambda \gg d$.

In this limit, the term that appears in both $\mathbf{A}_{M1}$ and $\mathbf{A}_{E2}$,

$$\left(\frac{1}{r} - ik\right) = -ik\left(1 + \frac{i}{kr}\right) \approx -ik \qquad \text{since } \frac{1}{kr} \sim \frac{\lambda}{r} \ll 1 \tag{6.2.39}$$

We can now compare the strengths of the different terms above that contribute to $\mathbf{A}_\omega$.

<u>Strengths of the different terms</u>

electric dipole:	$\mathbf{p}_\omega \sim qd$	$\mathbf{A}_{E1} \sim k\mathbf{p}_\omega \sim qkd$
magnetic dipole:	$\mathbf{m}_\omega \sim \dfrac{d\,j}{c} \sim \dfrac{dvq}{c}$	$\mathbf{A}_{M1} \sim k\mathbf{m}_\omega \sim qkd\left(\dfrac{v}{c}\right) \sim q(kd)^2$
electric quadrupole:	$\overleftrightarrow{\mathbf{Q}}_\omega \sim qd^2$	$\mathbf{A}_{E2} \sim k\dfrac{\omega}{c}\overleftrightarrow{\mathbf{Q}}_\omega \sim qd^2k\dfrac{\omega}{c} \sim q(kd)^2$

since $k = \omega/c$ and for the magnetic dipole term we used $v \sim d/\tau \sim \omega d \sim ckd$ to write $v/c \sim kd$.

Since by Approximation (2) we have $\lambda \gg d \;\Rightarrow\; kd \ll 1$, we see that with Approximations (1), (2) and (3), our expansion in r'/r is equivalent to an expansion in kd. The leading term in this expansion is thus the electric dipole term. The next order terms are the magnetic dipole and electric quadrupole,

$$\frac{|\mathbf{A}_{M1}|}{|\mathbf{A}_{E1}|} \sim \frac{|\mathbf{A}_{E2}|}{|\mathbf{A}_{E1}|} \sim kd \tag{6.2.40}$$

The next order terms, the magnetic quadrupole and the electric octupole, would by smaller than the electric dipole by a factor of $(kd)^2$.