Unit 6-3: Radiation in the Electric Dipole Approximation

In this section we focus on the electric dipole contribution to the radiation from an oscillating source. Within the *long wavelength* (non-relativistic) approximation, this is the leading term, provide the electric dipole moment $\mathbf{p}$ does not vanish.

In the last section we found that, within Approximations (1) and (2), the electric dipole contribution to the vector potential is,

$$\mathbf{A}_{E1}(\mathbf{r}) = -ik\mathbf{p}_\omega \frac{\mathrm{e}^{ikr}}{r} \qquad \text{where } k = \omega/c \tag{6.3.1}$$

Now we will find the fields $\mathbf{B}$ and $\mathbf{E}$.

Using $\boldsymbol{\nabla}\times(\phi\mathbf{F}) = (\boldsymbol{\nabla}\phi)\times\mathbf{F} + \phi\boldsymbol{\nabla}\times\mathbf{F}$ we have,

$$\mathbf{B}_{E1} = \boldsymbol{\nabla}\times\mathbf{A}_{E1} = -ik\left(\boldsymbol{\nabla}\frac{\mathrm{e}^{ikr}}{r}\right)\times\mathbf{p}_\omega \qquad \text{since } \mathbf{p}_\omega \text{ is a constant, then } \boldsymbol{\nabla}\times\mathbf{p}_\omega = 0 \tag{6.3.2}$$

$$= -ik\left(ik - \frac{1}{r}\right)\frac{\mathrm{e}^{ikr}}{r}\,\hat{\mathbf{r}}\times\mathbf{p}_\omega \tag{6.3.3}$$

$$= k^2\,\frac{\mathrm{e}^{ikr}}{r}\left(1 + \frac{i}{kr}\right)\hat{\mathbf{r}}\times\mathbf{p}_\omega \tag{6.3.4}$$

Adding Approximation (3), then in the *Radiation Zone*, we have $kr \gg 1$ and so,

$$\boxed{\mathbf{B}_{E1}(\mathbf{r}) = k^2\,\frac{\mathrm{e}^{ikr}}{r}\,\hat{\mathbf{r}}\times\mathbf{p}_\omega} \qquad \text{in the Radiation Zone} \tag{6.3.5}$$

To get the electric field, we use Ampere's law. Far from the source, where $\mathbf{j} = 0$, Ampere's law is,

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \tag{6.3.6}$$

For oscillating fields, $\mathbf{B}(\mathbf{r},t) = \mathbf{B}_\omega(\mathbf{r})\mathrm{e}^{-i\omega t}$ and $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_\omega\mathrm{e}^{-i\omega t}$, this becomes,

$$\boldsymbol{\nabla}\times\mathbf{B}_\omega = -\frac{i\omega}{c}\mathbf{E}_\omega \qquad \Rightarrow \qquad \mathbf{E}_{E1} = \frac{i}{k}\boldsymbol{\nabla}\times\mathbf{B}_{E1} \qquad \text{since } k = \omega/c \tag{6.3.7}$$

Therefore, within Approximations (1) and (2) we have,

$$\mathbf{E}_{E1} = \frac{i}{k}\,\boldsymbol{\nabla}\times\left[k^2\,\frac{\mathrm{e}^{ikr}}{r}\left(1+\frac{i}{kr}\right)\hat{\mathbf{r}}\times\mathbf{p}_\omega\right] \tag{6.3.8}$$

$$= \frac{i}{k}\left(\boldsymbol{\nabla}\mathrm{e}^{ikr}\right)\times\left[\frac{k^2}{r}\left(1+\frac{i}{kr}\right)\hat{\mathbf{r}}\times\mathbf{p}_\omega\right] + \frac{i}{k}\,\mathrm{e}^{ikr}\,\boldsymbol{\nabla}\times\left[\frac{k^2}{r}\left(1+\frac{i}{kr}\right)\hat{\mathbf{r}}\times\mathbf{p}_\omega\right] \tag{6.3.9}$$

Considering powers of $1/r$ in the above expression, the leading term is of order $1/r$. When we take the curl of the expression in the square brackets in the second term, the result is always of order $1/r^2$ (verify that for yourself!). In the Radiation Zone approximation, we keep only the leading $1/r$ term since the next order $1/r^2$ term will be a factor $1/(kr) \ll 1$ smaller.

Therefore, adding Approximation (3), we get in the Radiation Zone,

$$\mathbf{E}_{E1}(\mathbf{r}) = \left(\boldsymbol{\nabla}\mathrm{e}^{ikr}\right)\times\left[\frac{ik}{r}\,\hat{\mathbf{r}}\times\mathbf{p}_\omega\right] = ik\,\hat{\mathbf{r}}\,\mathrm{e}^{ikr}\times\left[\frac{ik}{r}\,\hat{\mathbf{r}}\times\mathbf{p}_\omega\right] \tag{6.3.10}$$

and so,

$$\boxed{\mathbf{E}_{E1}(\mathbf{r}) = -k^2\,\frac{e^{ikr}}{r}\,\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{p}_\omega)} \qquad \text{in the Radiation Zone} \tag{6.3.11}$$

If we had *not* made the Radiation Zone approximation, and worked out all the derivatives in Eq. (6.3.9), we would get,

$$\mathbf{E}_{E1}(\mathbf{r}) = k^2\,\frac{\mathrm{e}^{ikr}}{r}\,\left[\mathbf{p}_\omega - \hat{\mathbf{r}}\,(\mathbf{p}_\omega\cdot\hat{\mathbf{r}}) - \frac{i}{kr}\left(1+\frac{i}{kr}\right)\left(3\hat{\mathbf{r}}\,(\mathbf{p}_\omega\cdot\hat{\mathbf{r}}) - \mathbf{p}_\omega\right)\right] \tag{6.3.12}$$

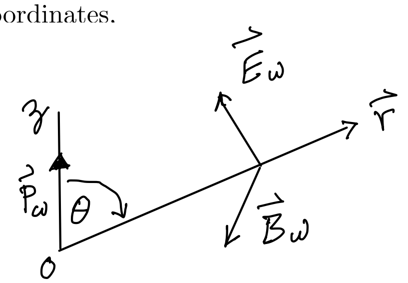
In the following, we will stick with the Radiation Zone approximation.

> **Discussion Question 6.3**
>
> Although in the following we will be interested in the Radiation Zone limit, we can also ask about the *Near Field* limit, where $d \ll r \ll \lambda$.
>
> The Near Field limit can be viewed as the limit where $kr \ll 1$, and so to leading order it will be the terms in $\mathbf{B}$ and $\mathbf{E}$ that have the highest power of $1/kr$ that dominate. What is the electric field, in the electric dipole approximation, in this Near Field limit? It should look familiar, can you recognize it? How does the ratio $|\mathbf{B}|/|\mathbf{E}|$ go? How does that compare with behavior in the Radiation Zone?

In the Radiation Zone,

$$\boxed{\begin{aligned}\mathbf{E}_{E1}(\mathbf{r}) &= -k^2\,\frac{e^{ikr}}{r}\,\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{p}_\omega)\\ \mathbf{B}_{B1}(\mathbf{r}) &= k^2\,\frac{e^{ikr}}{r}\,\hat{\mathbf{r}}\times\mathbf{p}_\omega\end{aligned}} \qquad \Rightarrow \qquad |\mathbf{E}_{E1}| = |\mathbf{B}_{E1}| \quad \text{and} \quad \mathbf{E}_{E1}\perp\mathbf{B}_{E1} \tag{6.3.13}$$

IF we assume that $\mathbf{p}_\omega$ is a real valued vector (later we will see cases where $\mathbf{p}_\omega$ is complex valued), then we can choose coordinates so that $\mathbf{p}_\omega = p_\omega\hat{\mathbf{z}}$ is aligned along the $\hat{\mathbf{z}}$ axis (if $\mathbf{p}_\omega$ were complex, then the real and imaginary parts could be in different directions, and so we could not align $\mathbf{p}_\omega$ along $\hat{\mathbf{z}}$). In that case we can write $\mathbf{E}_{E1}$ and $\mathbf{B}_{E1}$ in spherical coordinates.



$$\begin{aligned}\mathbf{E}_{E1}(\mathbf{r}) &= -k^2\,p_\omega\,\frac{e^{ikr}}{r}\,\sin\theta\,\hat{\boldsymbol{\theta}}\\ \mathbf{B}_{E1}(\mathbf{r}) &= -k^2\,p_\omega\,\frac{e^{ikr}}{r}\,\sin\theta\,\hat{\boldsymbol{\varphi}}\end{aligned} \tag{6.3.14}$$

To look at the power being radiated by the oscillating current, we now consider the Poynting vector,

$$\mathbf{S}_{E1}(\mathbf{r},t) = \frac{c}{4\pi}\mathrm{Re}\,\left[\mathbf{E}_{E1}(\mathbf{r})\mathrm{e}^{-i\omega t}\right]\times\mathrm{Re}\,\left[\mathbf{B}_{E1}(\mathbf{r})\mathrm{e}^{-i\omega t}\right] \tag{6.3.15}$$

Recall, it is crucial to take the real parts of the complex expressions, so as to get the real physical fields, *before* multiplying.

$$\mathrm{Re}\,\left[\mathbf{E}_{E1}(\mathbf{r})\mathrm{e}^{-i\omega t}\right] = -k^2\,p_\omega\,\frac{\cos(kr-\omega t)}{r}\,\sin\theta\,\hat{\boldsymbol{\theta}} \tag{6.3.16}$$

$$\mathrm{Re}\left[\mathbf{B}_{E1}(\mathbf{r})e^{-i\omega t}\right] = -k^2\, p_\omega, \frac{\cos(kr-\omega t)}{r}\, \sin\theta\, \hat{\boldsymbol{\varphi}} \tag{6.3.17}$$

So now,

$$\mathbf{S}_{E1}(\mathbf{r},t) = \frac{c}{4\pi}\, k^4\, p_\omega^2\, \frac{\cos^2(kr-\omega t)}{r^2}\, \sin^2\theta\, \hat{\mathbf{r}} \tag{6.3.18}$$

$\mathbf{S}_{E1} \sim \hat{\mathbf{r}} \quad \Rightarrow \quad$ the energy is flowing radially outwards.

$\mathbf{S}_{E1} \sim \dfrac{1}{r^2} \quad \Rightarrow \quad$ energy is conserved.

$\displaystyle\oint_S da\, \hat{\mathbf{n}} \cdot \langle \mathbf{S}_{E1} \rangle$ is constant when integrating over the surface of a sphere of any radius R.

But ...... if we did *not* make the Radiation Zone approximation, then the fields, and hence **S**, would have terms that went as higher powers of $1/r$. In particular, **S** would include terms that went like $1/r^3$, $1/r^4$, etc. Do these higher order terms mess up energy conservation? You will have to examine this in a homework problem!

Classically, we are usually only interested in the average energy current,

$$\langle \mathbf{S}_{E1} \rangle = \frac{1}{\tau}\int_0^\tau dt\, \mathbf{S}_{E1}(\mathbf{r},t) \qquad \text{where } \tau = 2\pi/\omega \text{ is the period of oscillation} \tag{6.3.19}$$

Using $\langle \cos^2(kr-\omega t)\rangle = 1/2$, we get,

$$\langle \mathbf{S}_{E1} \rangle = \frac{c}{8\pi}\, k^4\, p_\omega^2\, \frac{\sin^2\theta}{r^2}\, \hat{\mathbf{r}} \tag{6.3.20}$$

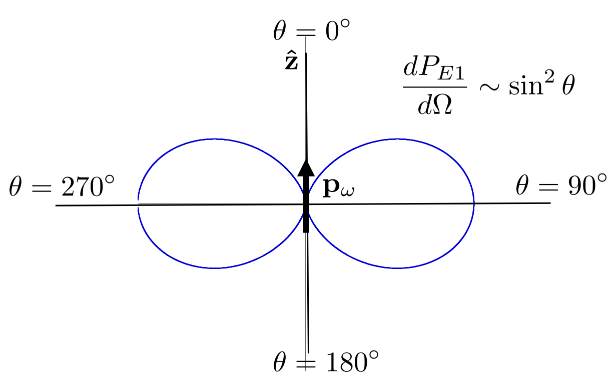
The average energy flowing through a differential element of area at spherical angles θ and φ is,

$$dP_{E1} = \hat{\mathbf{r}} \cdot \langle \mathbf{S}_{E1} \rangle\, r^2 \sin\theta\, d\theta\, d\varphi = \hat{\mathbf{r}} \cdot \langle \mathbf{S}_{E1} \rangle\, r^2\, d\Omega \tag{6.3.21}$$

where $d\Omega = \sin\theta\, d\theta\, d\varphi$ is the differential solid angle, and $r^2\, d\Omega$ is the differential surface area.

This then gives for the power cross-section

$$\frac{dP_{E1}}{d\Omega} = \hat{\mathbf{r}} \cdot \langle \mathbf{S}_{E1} \rangle\, r^2 = \frac{c}{8\pi}\, k^4\, p_\omega^2\, \sin^2\theta \sim \omega^4 \sin^2\theta \tag{6.3.22}$$



A polar plot of this distribution is shown on the left. The power distribution is rotationally symmetric about the $\hat{\mathbf{z}}$ axis, and so has the shape of a donut.

Most of the power is directed outwards into the xy plane $\perp$ $\mathbf{p}_\omega$, i.e. $dP_{E1}/d\Omega$ is peaked at 90°.

The total radiated power is,

$$P_{E1} = \int d\Omega \, \frac{dP_{E1}}{d\Omega} = \frac{c\,k^4\,p_\omega^2}{8\pi}\,2\pi \int_0^\pi d\theta\,\sin\theta\,\sin^2\theta \qquad \text{the integral} \quad \int_0^\pi d\theta \sin^3\theta = 4/3 \tag{6.3.23}$$

$$= \frac{c\,k^4\,p_\omega^2}{3} = \frac{p_\omega^2\,\omega^4}{3\,c^3} \sim \omega^4 \tag{6.3.24}$$

The power radiated goes like the fourth power of the frequency.

Why is the sky blue?

We can now give Lord Rayleigh's explanation for why the sky is blue!

When you look up at the sky, you are seeing the *indirect* light of the sun, i.e. the light emitted by the molecules in the atmosphere as they oscillate, and so radiate, due to the electric field they feel from the direct light of the sun.

The power of this radiated indirect light is $P \sim \omega^4\,p_\omega^2$.

Now $\mathbf{p} = \alpha\mathbf{E}$ with $\alpha \approx \dfrac{e^2}{m}\dfrac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$

For molecules in the atmosphere (the most common is N_2), the resonant frequency ω_0 for electronic excitations is typically higher than the visible spectrum, while for rotational excitations it is typically lower than the visible spectrum. So in the visible spectrum, $\alpha \approx$ constant, with little dependence on frequency ω. The dominant ω dependence of the power P is from the ω^4 factor.

So it is the light at higher frequencies that radiates the most.

Since light from the sun is "white light," it has components of all frequencies. Because of the ω^4 factor, of these different frequencies, it is the higher frequencies are scattered the most, and so they dominate the indirect light that we see.

Since it is blue that is the highest frequency in the visible spectrum, the indirect light we see when we look up at the sky is blue!

In contrast, when we look at a sunrise or sunset, we are looking into the sun and see the direct light. In this case what we see is dominated by those frequencies that are scattered the least. These are the lower frequencies. Hence sunrises and sunsets look red!