Unit 6-5: Radiation from Arbitrarily Time Varying Sources and Larmor's Formula

In the previous sections we considered the radiation from a pure harmonically oscillating source, i.e. oscillating at a single frequency ω. Here we consider a source with a general time dependence. We consider only the electric dipole approximation to the radiation, since in the long wavelength (non-relativistic) limit that is the leading term.

For $\mathbf{p}(t) = \mathbf{p}_\omega \mathrm{e}^{-i\omega t}$, a pure harmonic oscillation, we found that the radiated fields, in the electric dipole approximation, oscillate at the same frequency, $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_\omega(\mathbf{r})\mathrm{e}^{-i\omega t}$ and $\mathbf{B}(\mathbf{r},t) = \mathbf{B}_\omega(\mathbf{r})\mathrm{e}^{-i\omega t}$, with amplitudes given by,

$$\mathbf{E}_\omega = -k^2 \frac{\mathrm{e}^{ikr}}{r} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{p}_\omega) = -\frac{\omega^2}{c^2} \frac{\mathrm{e}^{i\omega r/c}}{r} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{p}_\omega) \tag{6.5.1}$$

$$\mathbf{B}_\omega = k^2 \frac{\mathrm{e}^{ikr}}{r} \hat{\mathbf{r}} \times \mathbf{p}_\omega = \frac{\omega^2}{c^2} \frac{\mathrm{e}^{i\omega r/c}}{r} \hat{\mathbf{r}} \times \mathbf{p}_\omega \qquad \text{since } k = \omega/c \tag{6.5.2}$$

For an arbitrarily time varying charge distribution, with electric dipole moment,

$$\mathbf{p}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathbf{p}_\omega \, \mathrm{e}^{-i\omega t} \tag{6.5.3}$$

the solution for the fields is obtained by superposition,

$$\mathbf{E}(\mathbf{r},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathbf{E}_\omega \, \mathrm{e}^{-i\omega t} = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\mathrm{e}^{-i\omega(t-r/c)}}{r} \left(\frac{\omega^2}{c^2}\right) \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{p}_\omega) \tag{6.5.4}$$

$$= \frac{-1}{c^2 r} \hat{\mathbf{r}} \times \left[\hat{\mathbf{r}} \times \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathrm{e}^{-i\omega(t-r/c)} \omega^2 \mathbf{p}_\omega\right] \tag{6.5.5}$$

$$= \frac{1}{c^2 r} \hat{\mathbf{r}} \times \left[\hat{\mathbf{r}} \times \frac{d^2}{dt^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathrm{e}^{-i\omega(t-r/c)} \mathbf{p}_\omega\right] \tag{6.5.6}$$

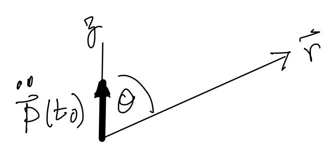
$$= \frac{1}{c^2 r} \hat{\mathbf{r}} \times \left[\hat{\mathbf{r}} \times \frac{d^2}{dt^2} \mathbf{p}(t-r/c)\right] \tag{6.5.7}$$

$$\boxed{\mathbf{E}(\mathbf{r},t) = \frac{1}{c^2 r} \hat{\mathbf{r}} \times [\hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t-r/c)]} \qquad \text{where} \quad \ddot{\mathbf{p}} = \frac{d^2\mathbf{p}}{dt^2} \tag{6.5.8}$$

define the *retarded time* $t_0 = t - r/c$.

In spherical coordinates, with $\ddot{\mathbf{p}}(t_0)$ aligned along $\hat{\mathbf{z}}$,

$$\mathbf{E}(\mathbf{r},t) = \frac{\ddot{p}(t_0)}{c^2 r} \sin\theta \, \hat{\boldsymbol{\theta}} \tag{6.5.9}$$



Similarly,

$$\mathbf{B}(\mathbf{r},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathbf{B}_\omega \, \mathrm{e}^{-i\omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\mathrm{e}^{-i\omega(t-r/c)}}{r} \left(\frac{\omega^2}{c^2}\right) \hat{\mathbf{r}} \times \mathbf{p}_\omega \tag{6.5.10}$$

$$= \frac{-1}{c^2 r} \hat{\mathbf{r}} \times \frac{d^2}{dt^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathrm{e}^{-i\omega(t-r/c)} \mathbf{p}_\omega \tag{6.5.11}$$

$$\boxed{\mathbf{B}(\mathbf{r},t) = \frac{-1}{c^2 r} \hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t_0)} \tag{6.5.12}$$

In the above spherical coordinates,

$$\mathbf{B}(\mathbf{r},t) = \frac{\ddot{p}(t_0)}{c^2\, r}\,\sin\theta\,\hat{\boldsymbol{\varphi}} \tag{6.5.13}$$

The Poynting vector is then,

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{c}{4\pi}\left(\frac{1}{c^2\, r}\right)^2 [\ddot{p}(t_0)]^2\,\sin^2\theta\,\hat{\mathbf{r}} \tag{6.5.14}$$

The total power radiated through the surface $\mathcal{S}$ of a sphere of radius r is then,

$$P = \oint_{\mathcal{S}} da\,\hat{\mathbf{n}}\cdot\mathbf{S} = 2\pi\int_0^\pi d\theta\,\sin\theta\, r^2\,\hat{\mathbf{r}}\cdot\mathbf{S} = \frac{[\ddot{p}(t_0)]^2}{2c^3}\int_0^\pi d\theta\,\sin^3\theta \tag{6.5.15}$$

Using $\int_0^\pi d\theta\,\sin^3\theta = 4/3$ we then get,

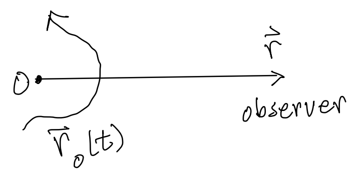
$$\boxed{P = \frac{2[\ddot{p}(t_0)]^2}{3c^3}} \tag{6.5.16}$$

Power radiated by an accelerating charge

Apply the above to a point charge q moving along the trajectory $\mathbf{r}_0(t)$. Then,

$$\mathbf{p}(t) = q\,\mathbf{r}_0(t) \qquad \text{so} \qquad \ddot{\mathbf{p}}(t) = q\,\ddot{\mathbf{r}}_0(t) = q\,\mathbf{a}(t)$$

where $\mathbf{a}$ is the charge's acceleration.



The radiated power is then,

$$\boxed{P = \frac{2}{3}\frac{q^2\,a^2(t_0)}{c^3}} \qquad \text{This is } \underline{\textit{Larmor's formula}} \text{ for the power radiated by an accelerating charge.} \tag{6.5.17}$$

The total power passing through a sphere of radius r at time t is due to the acceleration of the charge at the retarded time $t_0 = t - r/c$.

Note, there is only power radiated *if* the charge is accelerating! As we saw in Notes 5-1, a charge moving at constant velocity does *not* radiate.

power radiated $\sim$ (acceleration)^2

Since we derived this result using the electric dipole approximation in the long wavelength limit $\lambda \gg d$, the above Larmor's formula holds only in the limit of a *non-relativistically* moving charge, $v \ll c$. In unit 7 we will see how to extend Larmor's formula to the case where the charge may be moving relativistically fast.

Discussion Question 6.5

We derived the above expression for Larmor's formula by using the electric dipole approximation. The electric dipole approximation was the leading term in the long wavelength approximation, which can be viewed as giving an expansion in powers of $kq \sim v/c$.

If we want the analog of Larmor's formula, but now for a charge moving relativistically fast, so that v/c is not small, it suggests we would have to keep lots of higher order terms in this long wavelength approximation – we would have to keep the magnetic dipole term, the electric quadrupole term, the magnetic quadrupole term, the electric octupole term, and indeed all higher terms. Clearly we can't do that! So is there a clever way we could get the relativistic Larmor's formula without all that work?