Unit 7-4: Relativistic Larmor's Formula

In this, our final section, we discuss how to generalize the non-relativistic Larmor's formula of unit 6 to include the case when the charge is moving relativistically fast. The non-relativistic Larmor's formula was based on the electric dipole approximation in the long wavelength limit, $kd \ll 1$, which we saw was equivalent to the non-relativistic limit where all charges are moving with speeds $v \ll c$. Such an approximation is obviously poor when we have a charge moving with speed v approaching close to c. However this approximation is excellent, indeed exact, in the limit that the charge is instantaneously at rest with the charge having $\mathbf{v} = 0$. Note, even when the charge is *instantaneously* at rest, it still may be accelerating, and so radiating.

So our method will be to compute the power radiated in the charge's instantaneous rest frame where its $\mathring{\mathbf{v}} = 0$, which we can do using the non-relativistic Larmor's formula, and then do a Lorentz transformation of that power back to the original frame where the charge is moving with any velocity $\mathbf{v}$. We will use a circle above the symbol to denote quantities computed in the charge's instantaneous rest frame, such as $\mathring{\mathbf{v}} = 0$. [Note: since the charge is accelerating, the instantaneous rest frame at a time t will not be the same as the instantaneous rest frame at some other time t'.]

Consider the inertial frame of reference in which the charge is instantaneously at rest. We will call this frame $\mathring{\mathcal{K}}$. The power radiated in this frame is,

$$\mathring{P} = \frac{d\mathring{\mathcal{E}}}{d\mathring{t}} \tag{7.4.1}$$

where $\mathring{\mathcal{E}}$ is the energy radiated.

In $\mathring{\mathcal{K}}$ the momentum density of the radiated EM fields is $\mathring{\mathbf{\Pi}} = \frac{1}{4\pi c}\mathring{\mathbf{E}} \times \mathring{\mathbf{B}} \sim \mathring{\hat{\mathbf{r}}}$, which points outward in the radial direction. Integrating over all space, we see that contribution to the total EM momentum from oppositely oriented directions will cancel, and so the total radiated EM momentum vanishes, $\mathring{\mathcal{P}} = 0$.

We now consider the energy-momentum 4-vector giving the total energy and momentum of the radiated EM fields. This is,

$$\mathring{\mathcal{P}}_\mu = \left(\mathring{\mathcal{P}},\ \frac{i\mathring{\mathcal{E}}}{c}\right) \tag{7.4.2}$$

To get the radiated energy in the original frame $\mathcal{K}$, where the charge moves with velocity $\mathbf{v}$, we can make a Lorentz transformation of the energy-momentum 4-vector $\mathring{\mathcal{P}}_\mu$. Looking at the temporal component of that we get,

$$\frac{\mathcal{E}}{c} = \gamma\left(\frac{\mathring{\mathcal{E}}}{c} - \frac{\mathbf{v}}{c}\cdot\mathring{\mathcal{P}}\right) \qquad \Rightarrow \qquad \mathcal{E} = \gamma\,\mathring{\mathcal{E}} \qquad \text{since } \mathring{\mathcal{P}} = 0. \tag{7.4.3}$$

Also, taking the Lorentz transformation of the differential position of the charge $d\mathring{x}_\mu$, we get from the temporal component,

$$cdt = \gamma\left(cd\mathring{t} - \frac{\mathbf{v}}{c}\cdot d\mathring{\mathbf{r}}\right) \qquad \Rightarrow \qquad dt = \gamma\, d\mathring{t} \tag{7.4.4}$$

where the last step follows since, to lowest order, $d\mathring{\mathbf{r}} = 0$ since we are in the instantaneous rest frame, that is $\mathring{\mathbf{v}} = d\mathring{\mathbf{r}}/d\mathring{t} = 0$. Alternatively, since $\mathring{\mathcal{K}}$ is the instantaneous rest frame of the charge, $d\mathring{t}$ is the same as the proper time interval ds, and so $t = \gamma d\mathring{t}$ follows from Eq. (7.1.40).

So finally we have,

$$\frac{d\mathcal{E}}{dt} = \frac{\gamma d\mathring{\mathcal{E}}}{\gamma d\mathring{t}} = \frac{d\mathring{\mathcal{E}}}{d\mathring{t}} \qquad \Rightarrow \qquad P = \mathring{P} \tag{7.4.5}$$

The radiated power is a Lorentz invariant scalar!

In $\mathring{\mathcal{K}}$ we can use the non-relativistic Larmor's formula since $\mathring{\mathbf{v}} = 0$. So,

$$P = \frac{2}{3}\frac{q^2\mathring{a}^2}{c^3} \qquad \text{where } \mathring{a} \text{ is the acceleration of the charge in its rest frame } \mathring{\mathcal{K}}. \tag{7.4.6}$$

To write an expression for P without explicitly making mention of the rest frame $\mathring{\mathcal{K}}$, we need to find a Lorentz invariant scalar that reduces to a^2 as $\mathbf{v} \to 0$ (i.e. for one in the instantaneous rest frame). The only choice is α_μ^2, where α_μ is the 4-acceleration of the charge.

$$\alpha_\mu \;=\; \frac{du_\mu}{ds} \;=\; \gamma\,\frac{du_\mu}{dt} \;=\; \gamma\,\frac{d}{dt}\,(\gamma\mathbf{v},\; ic\gamma) \tag{7.4.7}$$

The spatial part is $\boldsymbol{\alpha} \;=\; \gamma^2\dfrac{d\mathbf{v}}{dt} + \gamma\mathbf{v}\dfrac{d\gamma}{dt},$ while the temporal part is $\alpha_4 = ic\gamma\dfrac{d\gamma}{dt}$

We have,

$$\frac{d\gamma}{dt} \;=\; \frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2/c^2}}\right) \;=\; \frac{\dfrac{\mathbf{v}}{c^2}\cdot\dfrac{d\mathbf{v}}{dt}}{(1-v^2/c^2)^{3/2}} \;=\; \frac{1}{c^2}\gamma^3\mathbf{v}\cdot\mathbf{a} \tag{7.4.8}$$

Now as $\mathbf{v} \to 0$, $\gamma \to 1$ and $d\gamma/dt \to 0$, so we have,

$$\boldsymbol{\alpha} \to \frac{d\mathbf{v}}{dt} = \mathbf{a} \qquad \text{and} \qquad \alpha_4 \to 0 \qquad\qquad \text{as } \mathbf{v} \to 0 \tag{7.4.9}$$

Thus,

$$\alpha_\mu \to (\mathbf{a},\; 0) \qquad \text{and so} \qquad \alpha_\mu^2 \to |\mathbf{a}|^2 \qquad \text{as } \mathbf{v} \to 0\text{, or equivalently, } \alpha_\mu^2 = \mathring{a}^2 \tag{7.4.10}$$

Thus we can write in any inertial frame of reference,

$$\boxed{P \;=\; \frac{2}{3}\frac{q^2}{c^3}\alpha_\mu^2 \;=\; \frac{2}{3}\frac{q^2}{c^3}\,(\mathring{u}_\mu)^2} \qquad \textit{the relativistic Larmor's formula} \tag{7.4.11}$$

4-Acceleration

From the above we have,

$$\alpha_\mu \;=\; \left(\gamma^2\frac{d\mathbf{v}}{dt} + \gamma\,\mathbf{v}\frac{d\gamma}{dt},\; ic\gamma\frac{d\gamma}{dt}\right) \tag{7.4.12}$$

Using $\dfrac{d\gamma}{dt} = \dfrac{1}{c^2}\gamma^3\mathbf{v}\cdot\mathbf{a}$ this becomes,

$$\alpha_\mu = \left(\gamma^2\mathbf{a} + \frac{1}{c^2}\gamma^4(\mathbf{v}\cdot\mathbf{a})\mathbf{v},\; \frac{ic\gamma^4}{c^2}\mathbf{v}\cdot\mathbf{a}\right) = \gamma^2\left(\mathbf{a} + \gamma^2\frac{(\mathbf{v}\cdot\mathbf{a})\mathbf{v}}{c^2},\; i\gamma^2\frac{\mathbf{v}\cdot\mathbf{a}}{c}\right) \tag{7.4.13}$$

and so,

$$\alpha_\mu^2 = \gamma^4\left[a^2 + 2\gamma^2\,\frac{(\mathbf{v}\cdot\mathbf{a})^2}{c^2} + \gamma^4\frac{(\mathbf{v}\cdot\mathbf{a})^2v^2}{c^4} - \gamma^4\frac{(\mathbf{v}\cdot\mathbf{a})^2}{c^2}\right] = \gamma^2\left[a^2 + \gamma^2\,\frac{(\mathbf{v}\cdot\mathbf{a})^2}{c^2}\left(2 + \gamma^2\,\frac{v^2}{c^2} - \gamma^2\right)\right] \tag{7.4.14}$$

Now $\gamma^2(1 - v^2/c^2) = \gamma^2/\gamma^2 = 1$, so the term in the big parenthesis on the right is just $(2-1) = 1$, and we get,

$$\boxed{\alpha_\mu^2 = \gamma^4\left[a^2 + \gamma^2\,\frac{(\mathbf{v}\cdot\mathbf{a})^2}{c^2}\right]} \tag{7.4.15}$$

which is just the square of the charge's acceleration in the charge's instantaneous rest frame, $\mathring{a}^2$.

Linear Motion

For a charge accelerating in *linear* motion, velocity and acceleration are co-linear, and so we have $\mathbf{v} \cdot \mathbf{a} = va$, and therefore,

$$\alpha_\mu^2 = \gamma^4\, a^2 \left[1 + \gamma^2\, \frac{v^2}{c^2}\right] \tag{7.4.16}$$

Using,

$$\gamma^2 = \frac{1}{1 - v^2/c^2} \qquad \Rightarrow \qquad \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} \qquad \Rightarrow \qquad 1 + \gamma^2\, \frac{v^2}{c^2} = \gamma^2 \tag{7.4.17}$$

we get

$$\alpha_\mu^2 = \gamma^6\, a^2 \qquad \text{for linear motion} \tag{7.4.18}$$

So,

$$P = \frac{2}{3} \frac{q\, a^2}{c^3}\, \gamma^6 \tag{7.4.19}$$

For linear motion, the radiated power is increased by a factor γ^6 compared to the non-relativistic result.

Circular Motion

For a charge moving in circular motion, $\mathbf{v} \cdot \mathbf{a} = 0$, and we therefore have $\alpha_\mu^2 = \gamma^4\, a^2$. The radiated power is then,

$$P = \frac{2}{3} \frac{q\, a^2}{c^3}\, \gamma^4 \tag{7.4.20}$$

For a charge moving in circular motion, the radiated power is increased by a factor γ^4 compared to the non-relativistic result.