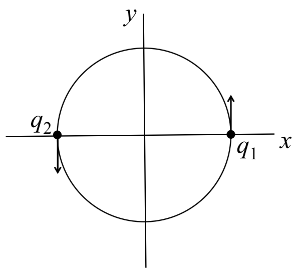
Consider two charges q_1 and q_2 moving on opposite sides of a circular path of radius d, centered on the origin, orbiting with angular frequency ω. We have $\mathbf{r}_2 = -\mathbf{r}_1$, with $\mathbf{r}_1(t) = \hat{\mathbf{x}}\, d\cos(\omega t) + \hat{\mathbf{y}}\, d\sin(\omega t)$.



We computed the electric dipole moment and found $\mathbf{p} \propto (q_1 - q_2)$, while the electric quadrupole moment was $\mathbf{Q}' \propto (q_1 + q_2)$.

Consider two cases:

(i) $q_1 = -q_2 \equiv q$ and (ii) $q_1 = q_2 \equiv q$

In case (i) we showed that there is a non-zero amplitude $\mathbf{p}_\omega$ for the oscillating electric dipole moment $\mathbf{p}(t)$ and that the electric quadrupole moment $\mathbf{Q}' = 0$.

In case (ii) we showed that $\mathbf{p} = 0$, but that the oscillation of $\mathbf{Q}'$ is non-zero, moreover $\mathbf{Q}'$ oscillates with a frequency 2ω rather than frequency ω. We explained the oscillation with 2ω by noting that when $q_1 = q_2$, then after one half period of orbit, q_1 and q_2 have exchanged places, and since $q_1 = q_2$ the system now looks exactly the same as at the start of the period. The charge density therefore oscillates with a period half that of the period of the charges orbit, and so the frequency is 2ω.

What happens if $q_1 \neq q_2$? In that case, since $\mathbf{p}_\omega \propto (q_1 - q_2)$ there will be an electric dipole component to the radiation at frequency ω. But because $\mathbf{Q}'_{2\omega} \propto (q_1 + q_2)$, there will also be an electric quadrupole component to the radiation at frequency 2ω.

Since $q_1 \neq q_2$, the charge configuration does not return to itself after one half period of the charges' orbit, so how do we physically explain the component of the radiation at frequency 2ω?

We can define,

$$\bar{q} = \frac{q_1 + q_2}{2} \quad \text{and} \quad \delta q = \frac{q_1 - q_2}{2} \qquad \text{so that} \qquad q_1 = \bar{q} + \delta q \quad \text{and} \quad q_2 = \bar{q} - \delta q \tag{1}$$

Consider the charge density $\rho(\mathbf{r}, t)$ associated with q_1 and q_2 to be made up of two pieces, $\rho = \bar{\rho} + \delta\rho$, with,

$$\bar{\rho}(\mathbf{r}, t) = \bar{q}\,\delta(\mathbf{r} - \mathbf{r}_1(t)) + \bar{q}\,\delta(\mathbf{r} + \mathbf{r}_1(t)) \qquad \text{and} \qquad \delta\rho(\mathbf{r}, t) = \delta q\,\delta(\mathbf{r} - \mathbf{r}_1(t)) - \delta q\,\delta(\mathbf{r} + \mathbf{r}_1(t)) \tag{2}$$

so that,

$$\rho = \bar{\rho} + \delta\rho = q_1\,\delta(\mathbf{r} - \mathbf{r}_1(t)) + q_2\,\delta(\mathbf{r} + \mathbf{r}_1(t)) \tag{3}$$

Because the multipole moments are linear in ρ, then the electric dipole moment and electric quadrupole moment are just linear combinations of the contributions from $\bar{\rho}$ and from $\delta\rho$.

The distribution $\delta\rho$ is just like case (i), and so it therefore has an electric dipole moment that oscillates at frequency ω, and so gives a component to the radiation at frequency ω. But the electric quadrupole moment from $\delta\rho$ vanishes.

The distribution $\bar{\rho}$ is just like case (ii), and so it has a vanishing electric dipole moment, but it has a finite electric quadrupole moment that oscillates with frequency 2ω, since the charges that make up $\bar{\rho}$ do indeed change places after only one half period of orbit. This part of the charge distribution therefore gives rise to an electric quadrupole component to the radiation which oscillates at frequency 2ω.