

Unit 1: From Coulomb to Maxwell

Electrodynamics is concerned with one particular attribute of matter – *charge*. In this unit we will review the development of Maxwell’s equations for classical electrodynamics, in terms of both fields and potentials. We will understand the different sets of units that are commonly used (in this course we will use CGS units). We will discuss the gauge invariance of potentials and how this leads to different representations of the potentials. We will see that waves are a solution. We will review Fourier transforms, that play an important role in simplifying many calculations and in understanding many behaviors.

Unit 1-1: Electrostatics

Experimentally it was observed that certain bodies exert long range forces on each other that are certainly not gravitational – they are not proportional to the body’s mass and they can be repulsive as well as attractive. The source of this new force was defined to be the *charge* carried by the body.

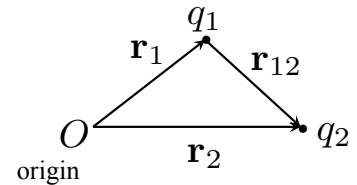
Coulomb’s Law

For a charge q_1 at position \mathbf{r}_1 and charge q_2 at position \mathbf{r}_2 , if the separation between the charges $|\mathbf{r}_2 - \mathbf{r}_1|$ is much greater than the size of either charge, then the force *on* charge 2 due to the presence of charge 1 is given by Coulomb’s Law.

$$\mathbf{F}_{12} = k_1 q_1 q_2 \frac{\hat{\mathbf{r}}_{12}}{r_{12}^2} \quad \text{where} \quad \mathbf{r}_{12} \equiv \mathbf{r}_2 - \mathbf{r}_1 \quad \text{and} \quad \hat{\mathbf{r}}_{12} \equiv \frac{\mathbf{r}_{12}}{|\mathbf{r}_{12}|} \quad \text{is the unit vector pointing from 1 to 2} \quad (1.1.1)$$

\mathbf{F}_{12} is a central force – it points from 1 to 2. It is an inverse square law $\mathbf{F}_{12} \propto 1/|\mathbf{r}_{12}|^2$. It is repulsive if $q_1 q_2 > 0$, and attractive if $q_1 q_2 < 0$.

k_1 is a *universal* constant of nature that determines the strength of the force when q is expressed in terms of some given reference charge.



Since we only know about charge by measuring the Coulomb force that one charge exerts on another charge, we are in principle free to choose k_1 to be anything we like – our choice then determines the units that charge is measure in.

In the MKS system of units (also called SI units) charge is measured in the historical unit, the *coulomb*. Then k_1 has the value $k_1 = 1/(4\pi\epsilon_0) = 10^{-7}c^2$, where c is the speed of light in a vacuum. The units of k_1 are $Nt \cdot m^2/coul^2$.

In the CGS system of units (also called esu – electrostatic units) one fixes $k_1 = 1$ and charge is measured in *statcoulombs*. k_1 is taken as dimensionless, so the units of the *statcoulomb* is the same as $(dyne - cm^2)^{1/2}$.

Another reasonable modern choice would be to measure charge in integer multiples of the electron charge. This would yield a different value for k_1 .

In this class we will be using CGS units, but for now we will keep k_1 as a general constant, to be fixed later.

Superposition

For charges q_i at positions \mathbf{r}_i , the force on a charge Q at position \mathbf{r} is

$$\mathbf{F} = k_1 Q \sum_i q_i \frac{(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3} \quad \text{forces add linearly} \quad \left(\text{note, } \frac{\hat{\mathbf{r}}}{|\mathbf{r}|^2} = \frac{\mathbf{r}}{|\mathbf{r}|^3} \right) \quad (1.1.2)$$

Conservation of Charge

Charge is neither created nor destroyed

$$\frac{d}{dt} \sum_i q_i = 0, \quad \text{where the sum is over all charges in the system} \quad (1.1.3)$$

Continuum Charge Density

For charges q_i at positions \mathbf{r}_i , we define the continuum charge density

$$\rho(\mathbf{r}) = \sum_i q_i \delta(\mathbf{r} - \mathbf{r}_i) \quad (1.1.4)$$

$\delta(\mathbf{r} - \mathbf{r}_i)$ is the Dirac delta function, which has the properties when integrating over some volume V ,

$$\int_V d^3r \delta(\mathbf{r} - \mathbf{r}_i) = \begin{cases} 1 & \text{if } \mathbf{r}_i \in V \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \int_V d^3r f(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}_i) = \begin{cases} f(\mathbf{r}_i) & \text{if } \mathbf{r}_i \in V \\ 0 & \text{otherwise} \end{cases} \quad (1.1.5)$$

where $f(\mathbf{r})$ is any smooth scalar function. With this definition of $\rho(\mathbf{r})$ we have,

$$\int_V d^3r \rho(\mathbf{r}) = \sum_i q_i \int_V d^3r \delta(\mathbf{r} - \mathbf{r}_i) = \sum_{\mathbf{r}_i \in V} q_i = Q_{\text{encl}} \quad (1.1.6)$$

where Q_{encl} is the total charge enclosed within the volume V .

Since $\rho(\mathbf{r})$ has the units of charge per volume, it follows that $\delta(\mathbf{r})$ must have the units of 1/volume. In general, $\delta(x)$ has units that are equal to $1/(\text{units of } x)$.

We can now write the Coulomb force in terms of the continuum charge density ρ . The force on a charge Q at position \mathbf{r} , due to other charges describe by the charge density $\rho(\mathbf{r})$ is given by,

$$\mathbf{F} = k_1 Q \int d^3r' \rho(\mathbf{r}') \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \quad (1.1.7)$$

If one substitutes the expression for ρ in Eq. (1.1.4) into the above, and integrates over the delta functions, one regains our original form of Eq. (1.1.2).

We will generally “forget” that $\rho(\mathbf{r})$ is in principle, at the microscopic level, made up of a distribution of point charges, and treat $\rho(\mathbf{r})$ as if it was a smooth continuous function.

In terms of $\rho(\mathbf{r})$, charge conservation can be written as

$$\frac{d}{dt} \left[\int_V d^3r \rho(\mathbf{r}) \right] = 0 \quad (1.1.8)$$

assuming that V is so large that it contains all the charge, and that no charge flows through the surface of V .

Electric Field and Maxwell's Equations for Electrostatics

The electric field $\mathbf{E}(\mathbf{r})$ is the force per unit charge that would be felt by an infinitesimal test charge δq at position \mathbf{r} ,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{\delta q} \mathbf{F} = k_1 \int d^3r' \rho(\mathbf{r}') \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \quad (1.1.9)$$

In principle, the above is the solution to *all* electrostatic problems. If one knows $\rho(\mathbf{r})$, then the above lets one determine the resulting $\mathbf{E}(\mathbf{r})$. In practice, however, we may not always know what is $\rho(\mathbf{r})$, but may have to solve for ρ self-consistently with \mathbf{E} .

• Discussion Question 1.1

Can you think of a situation in which we would not apriori know ρ , and so not be able to apply Eq. (1.1.9) directly?

(1.1.10)

It will therefore help to have another formulation of Coulomb's law in terms of differential equations. We get this by taking the divergence and the curl of Eq. (1.1.9) to get (proof to follow later!) Maxwell's equations for electrostatics,

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = 4\pi k_1 \rho(\mathbf{r}) \quad \text{(a) Gauss' Law}$$

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0 \quad \text{(b) true *only* for statics!}$$

(1.1.11)

If we take the above differential equations as given, we can then recast them into the integral form in which you probably first encountered them.

By Gauss' Theorem of vector calculus,

$$\oint_S da \hat{\mathbf{n}} \cdot \mathbf{E} = \int_V d^3r \nabla \cdot \mathbf{E} = 4\pi k_1 \int_V d^3r \rho = 4\pi k_1 Q_{\text{encl}} \quad (1.1.12)$$

where S is the surface bounding the volume V , da is the differential area on the surface S , and $\hat{\mathbf{n}}$ is the outward pointing unit normal vector on S , and Q_{encl} is the total charge enclosed in V .

By Stoke's Theorem of vector calculus,

$$\oint_C d\boldsymbol{\ell} \cdot \mathbf{E} = \int_S da \hat{\mathbf{n}} \cdot \nabla \times \mathbf{E} = 0 \quad (1.1.13)$$

where C is the curve bounding the surface S and $d\boldsymbol{\ell}$ is the differential length pointing tangent to C .

Maxwell's equations for electrostatics in integral form are therefore

$$\oint_S da \hat{\mathbf{n}} \cdot \mathbf{E} = 4\pi k_1 Q_{\text{encl}} \quad \oint_C d\boldsymbol{\ell} \cdot \mathbf{E} = 0 \quad (1.1.14)$$

We now prove that (a) and (b) above follow from the Coulomb expression of Eq. (1.1.9).

First we note an important result that is *very* good to remember

$$\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = -\nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \quad (1.1.15)$$

To see this, let $\mathbf{s} = \mathbf{r} - \mathbf{r}'$, and do the calculation in spherical coordinates centered at $\mathbf{s} = 0$.

$$\text{Then} \quad \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{\hat{\mathbf{s}}}{s^2} \quad \text{and} \quad -\nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\nabla \left(\frac{1}{s} \right) \quad (1.1.16)$$

Since $\mathbf{s} = \mathbf{r} - \mathbf{r}'$, then the differential operator ∇ (which differentiates with respect to \mathbf{r}) is the same as ∇_s (which differentiates with respect to \mathbf{s}). In spherical coordinates (s, θ, φ) one has

$$\nabla f = \hat{\mathbf{s}} \frac{\partial f}{\partial s} + \hat{\boldsymbol{\theta}} \frac{1}{s} \frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\varphi}} \frac{1}{s \sin \theta} \frac{\partial f}{\partial \varphi} \quad (1.1.17)$$

$$\text{So} \quad -\nabla \left(\frac{1}{s} \right) = -\hat{\mathbf{s}} \frac{d}{ds} \left(\frac{1}{s} \right) = \frac{\hat{\mathbf{s}}}{s^2} \quad (1.1.18)$$

where in the first step we used $\nabla = \hat{\mathbf{s}}(d/ds)$ in spherical coordinates, since the function $1/s$ depends only on the radial coordinate s and not the orientations θ and φ . Substituting back in for \mathbf{s} gives the desired Eq. (1.1.15).

Using Eq. (1.1.15) we can then write,

$$\mathbf{E}(\mathbf{r}) = k_1 \int d^3 r' \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = -k_1 \int d^3 r' \rho(\mathbf{r}') \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\nabla \left(k_1 \int d^3 r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) \quad (1.1.19)$$

where in the last step we can pull the ∇ outside the integral because ∇ acts on the variable \mathbf{r} , while the integral is over the variable \mathbf{r}' . We thus find that \mathbf{E} is the gradient of a scalar function. This leads us directly to conclude,

$$\boxed{\nabla \times \mathbf{E} = 0} \quad (b) \quad (1.1.20)$$

since the curl of a gradient always vanishes, $\nabla \times \nabla f = 0$ for any scalar function f .

Next we consider

$$\nabla \cdot \mathbf{E} = -\nabla^2 \left(k_1 \int d^3 r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) = -k_1 \int d^3 r' \rho(\mathbf{r}') \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \quad \text{where} \quad \nabla^2 = \nabla \cdot \nabla \quad (1.1.21)$$

Consider the last term involving the Laplacian operator ∇^2 . Again we write $\mathbf{s} = \mathbf{r} - \mathbf{r}'$, $\nabla = \nabla_s$, and evaluate in spherical coordinates with the origin at $s = 0$. The Laplacian in spherical coordinates is

$$\nabla^2 f = \frac{1}{s^2} \frac{\partial}{\partial s} \left(s^2 \frac{\partial f}{\partial s} \right) + \frac{1}{s^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{s^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2} \quad (1.1.22)$$

so applying to $f = 1/s$ gives,

$$\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = \nabla^2 \left(\frac{1}{s} \right) = \frac{1}{s^2} \frac{d}{ds} \left[s^2 \frac{d}{ds} \left(\frac{1}{s} \right) \right] = \frac{1}{s^2} \frac{d}{ds} [-1] = \begin{cases} 0 & \text{for } s \neq 0 \\ \text{undefined} & \text{at } s = 0 \end{cases} \quad (1.1.23)$$

Thus $\nabla^2(1/s)$ vanishes everywhere except maybe at $s = 0$. To see what happens at $s = 0$, consider integrating over a sphere V of radius R centered at the origin.

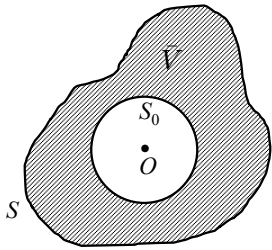
$$\int_V d^3 s \nabla^2 \left(\frac{1}{s} \right) = \int_V d^3 s \nabla \cdot \nabla \left(\frac{1}{s} \right) = \oint_S da \hat{\mathbf{n}} \cdot \nabla \left(\frac{1}{s} \right) \quad \text{using Gauss' Theorem in the last step.} \quad (1.1.24)$$

Since the surface S is a sphere, the outward normal is just the same as the unit vector in the radial direction, $\hat{\mathbf{n}} = \hat{\mathbf{s}}$. So

$$\hat{\mathbf{n}} \cdot \nabla \left(\frac{1}{s} \right) = \hat{\mathbf{s}} \cdot \hat{\mathbf{s}} \frac{d}{ds} \left(\frac{1}{s} \right) = -\frac{1}{s^2} \quad \text{which is constant everywhere on the surface of the sphere } S. \quad (1.1.25)$$

Since $s = R$ on the surface of the sphere we then have

$$\oint_S da \hat{\mathbf{n}} \cdot \nabla \left(\frac{1}{s} \right) = -\oint da \left(\frac{1}{R^2} \right) = -4\pi R^2 \left(\frac{1}{R^2} \right) = -4\pi \quad (1.1.26)$$



The above was for integrating over a sphere centered at the origin, but we can show that we get the same result if we integrate over *any* volume V that contains the origin $\mathbf{s} = 0$. Let S be any surface that encloses the origin. There is always some sphere with surface S_0 that encloses the origin and lies entirely within the surface S . Let \bar{V} be the surface bounded by S on the outside, and by S_0 on the inside, so that \bar{V} excludes the origin. \bar{V} is bounded by the surface $S \cup S_0$. Then by Gauss' Theorem we have

$$\int_{\bar{V}} d^3s \nabla^2 \left(\frac{1}{s} \right) = \int_{\bar{V}} d^3s \nabla \cdot \nabla \left(\frac{1}{s} \right) = \oint_{S \cup S_0} da \hat{\mathbf{n}} \cdot \nabla \left(\frac{1}{s} \right) = \oint_S da \hat{\mathbf{n}} \cdot \nabla \left(\frac{1}{s} \right) - \oint_{S_0} da \hat{\mathbf{n}} \cdot \nabla \left(\frac{1}{s} \right) \quad (1.1.27)$$

where, in the two integrals on the right, $\hat{\mathbf{n}}$ is pointing in the outward radial direction, $\hat{\mathbf{n}} = \hat{\mathbf{s}}$.

Now we know that $\nabla^2(1/s) = 0$ everywhere in \bar{V} , since $\nabla^2(1/s)$ is only nonzero at $\mathbf{s} = 0$, which is outside \bar{V} . Therefore,

$$\int_{\bar{V}} d^3s \nabla^2 \left(\frac{1}{s} \right) = 0 \Rightarrow \oint_S da \hat{\mathbf{n}} \cdot \nabla \left(\frac{1}{s} \right) = \oint_{S_0} da \hat{\mathbf{n}} \cdot \nabla \left(\frac{1}{s} \right) = -4\pi \quad (1.1.28)$$

So now if V is the volume bounded by S (and so includes the origin), we have by Gauss' Theorem,

$$\int_V d^3s \nabla^2 \left(\frac{1}{s} \right) = \oint_S da \hat{\mathbf{n}} \cdot \nabla \left(\frac{1}{s} \right) = -4\pi \quad (1.1.29)$$

from which we conclude that the integral of $\nabla^2(1/s)$, over any volume that includes the origin, is just -4π . So we have concluded that $\nabla^2(1/s)$ is zero everywhere except at the origin $\mathbf{s} = 0$, but its integral over any volume including the origin is -4π . These are exactly the same properties as the Dirac delta function. So we conclude,

$$\nabla^2 \left(\frac{1}{s} \right) = -4\pi\delta(\mathbf{s}) \Rightarrow \boxed{\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi\delta(\mathbf{r} - \mathbf{r}')} \quad (1.1.30)$$

This is an important result that is *very* good to remember!

We now return to Eq. (1.1.21) where we can use the property of the delta function to write

$$\boxed{\nabla \cdot \mathbf{E} = -k_1 \int d^3r' \rho(r') \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -k_1 \int d^3r' \rho(r') (-4\pi)\delta(\mathbf{r} - \mathbf{r}') = 4\pi k_1 \rho(\mathbf{r})} \quad (a) \quad (1.1.31)$$

We have seen that the Coulomb-like expression for \mathbf{E} in Eq. (1.1.9) leads to the differential Maxwell's equations for electrostatics, Eq. (1.1.11). We can ask if the reverse is true. If start with the formulation in terms of the differential equations Eq. (1.1.11), does that lead to the Coulomb-like solution of Eq. (1.1.9)? Are the two formulations completely equivalent. The answer is *yes*, because of Helmholtz's Theorem.

Helmholtz's Theorem of vector calculus: If one specifies both the diverge *and* curl of a vector function, *and* specifies suitable boundary conditions (here $\mathbf{E} \rightarrow 0$ as $r \rightarrow \infty$ and one is away from all charges), then the vector function is *uniquely* determined.

Helmholtz's Theorem

Suppose the divergence and curl of a vector field are given, and the value of the field is specified on the surface of the system, that is,

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = f(\mathbf{r}) \quad \text{and} \quad \nabla \times \mathbf{E}(\mathbf{r}) = \mathbf{g}(\mathbf{r}) \quad \text{for } \mathbf{r} \text{ within a volume } V \quad (1.1.32)$$

and

$$\mathbf{E}(\mathbf{r}) = \mathbf{h}(\mathbf{r}) \quad \text{for } \mathbf{r} \text{ on the surface } S \text{ of the volume } V, \quad (1.1.33)$$

where $f(\mathbf{r})$, $\mathbf{g}(\mathbf{r})$, and $\mathbf{h}(\mathbf{r})$ are given specified functions. Then this information will uniquely determine the vector field $\mathbf{E}(\mathbf{r})$.

Proof: Suppose there were two different solutions, $\mathbf{E}(\mathbf{r})$ and $\mathbf{E}'(\mathbf{r})$. Then define their difference $\mathbf{G}(\mathbf{r}) = \mathbf{E}(\mathbf{r}) - \mathbf{E}'(\mathbf{r})$. $\mathbf{G}(\mathbf{r})$ must then satisfy,

$$\nabla \cdot \mathbf{G}(\mathbf{r}) = 0 \quad \text{and} \quad \nabla \times \mathbf{G}(\mathbf{r}) = 0 \quad \text{for all } \mathbf{r} \text{ in the volume } V, \quad (1.1.34)$$

and

$$\mathbf{G}(\mathbf{r}) = 0 \quad \text{for all } \mathbf{r} \text{ on the surface } S. \quad (1.1.35)$$

Now the condition $\nabla \times \mathbf{G} = 0$ implies that we can find a scalar function $\phi(\mathbf{r})$ such that $\mathbf{G}(\mathbf{r}) = \nabla\phi(\mathbf{r})$. Then the condition $\nabla \cdot \mathbf{G} = 0 \Rightarrow \nabla^2\phi = 0$ for all \mathbf{r} in V . We will show that this implies that $\phi(\mathbf{r})$ must be a constant.

A function ϕ that satisfies $\nabla^2\phi = 0$ within a region V is said to be a *harmonic* function on V . An important property of harmonic functions, that you will prove later for homework, is that the value of the function at a position \mathbf{r} is equal to the average of the values of the function on the surface of a sphere centered at position \mathbf{r} (assuming the sphere lies entirely within the volume V),

$$\phi(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{S_0} da' \phi(\mathbf{r}') \quad \text{where } S_0 \text{ is the surface of a sphere of radius } R, \text{ centered at the point } \mathbf{r}. \quad (1.1.36)$$

From this property we can conclude that a harmonic function on V can have no local maximum or minimum within the volume V . All maxima or minima must lie on the surface S bounding the volume V . This follows from the following argument. Consider a small sphere of radius R centered about the point \mathbf{r} such that the sphere fits within the volume V . If \mathbf{r} was a local maximum, and if the radius of the sphere is sufficiently small, it must be the case that for \mathbf{r}' on the surface of the sphere, $\phi(\mathbf{r}') < \phi(\mathbf{r})$. But then, integrating over the surface of this sphere, we would have $(1/4\pi R^2) \oint da' \phi(\mathbf{r}') < \phi(\mathbf{r})$. This would violate the above property of a harmonic function, so we must conclude that \mathbf{r} cannot be a local maximum. By a similar argument we can conclude that \mathbf{r} cannot be a local minimum. Hence all local maxima and minima must lie on the surface of V .

Back to the function $\mathbf{G}(\mathbf{r})$ defined above. We have $\nabla \cdot \mathbf{G} = 0$ and $\mathbf{G} = \nabla\phi$ which implies $\nabla^2\phi = 0$ everywhere in V . So ϕ is an harmonic function. But we also have that $\mathbf{G}(\mathbf{r}) = \nabla\phi(\mathbf{r}) = 0$ for all \mathbf{r} on the surface S . So we conclude that $\phi(\mathbf{r})$ must be constant on the surface S , i.e., $\phi(\mathbf{r}) = C$, with C a constant, for all \mathbf{r} on S .

Since ϕ is an harmonic function, it must be the case that all maxima and minima of ϕ over the volume V lie on the surface S . Since $\phi(\mathbf{r})$ is constant everywhere on the surface S it must be true that the maximum value of ϕ and the minimum value of ϕ over the volume V must both be equal to this constant, $\phi_{\max} = \phi_{\min} = C$. From this it follows that we must have $\phi(\mathbf{r}) = C$ for all \mathbf{r} inside V (because if not, there would then be a point in V where ϕ was either bigger or smaller than C , and that would not be consistent with C being both the maximum and the minimum of ϕ).

Since $\phi(\mathbf{r}) = C$ is a constant everywhere in V , then $\nabla\phi(\mathbf{r}) = \mathbf{G}(\mathbf{r}) = 0$ everywhere in V . Hence $\mathbf{E}(\mathbf{r}) = \mathbf{E}'(\mathbf{r})$ for all \mathbf{r} in V . The solution is unique!