Unit 1-3-S: Supplementary Material - Faraday's Law Revisited: Motional emf and special relativity

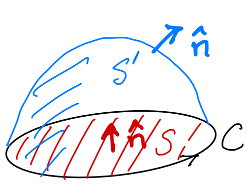
We introduced Faraday's Law of Induction in the previous section in a particular limited form:

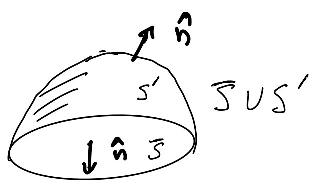
$$\oint_C \boldsymbol{d\ell}\cdot\mathbf{E} = -\frac{1}{c}\frac{d}{dt}\left[\int_S da\,\hat{\mathbf{n}}\cdot\mathbf{B}\right] \qquad \text{here we will use CGS units} \tag{1.3.S.1}$$

Here C is a *fixed* (i.e. not moving in time) curve in space and S is a surface bounded by that curve. The left hand side is called the electromotive force around the loop, and is denoted as the emf or $\mathcal{E}$. The right hand side is proportional to the time derivative of $\int_S \hat{\mathbf{n}}\cdot\mathbf{B} \equiv \Phi$, the magnetic flux through the surface S.

One can ask, does it matter which surface S on chooses to evaluate the flux on the right hand side? Suppose we have two surfaces S and S', both bounded by the same curve C. Then one can show that, because $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, then the magnetic flux must be the same, i.e. $\Phi = \int_S da\,\hat{\mathbf{n}}\cdot\mathbf{B} = \int_{S'} da\,\hat{\mathbf{n}}\cdot\mathbf{B}$.

To see this, suppose for example C is a circle in the xy plane. Let S be the surface enclosed by the circle in the xy plane, while S' is the northern hemisphere of a sphere whose equator is C, as illustrated below. If we define $\bar{S}$ as the same as the surface S, but with the direction of the normal to the surface $\hat{\mathbf{n}}$ with reversed sign, then $\bar{S}\cup S'$ forms a closed surface. Using Gauss' Theorem of vector calculus, and the Maxwell equation $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, we then have that the flux through $\bar{S}\cup S'$ must vanish,





$$\int_{\bar{S}\cup S'} da\,\hat{\mathbf{n}}\cdot\mathbf{B} = \int_V d^3r\,\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \qquad \text{where } V \text{ is the volume bounded by the surface } \bar{S}\cup S' \tag{1.3.S.2}$$

But we can then write,

$$\int_{\bar{S}\cup S'} da\,\hat{\mathbf{n}}\cdot\mathbf{B} = \int_{S'} da\,\hat{\mathbf{n}}\cdot\mathbf{B} + \int_{\bar{S}} da\,\hat{\mathbf{n}}\cdot\mathbf{B} = \int_{S'} da\,\hat{\mathbf{n}}\cdot\mathbf{B} - \int_{S} da\,\hat{\mathbf{n}}\cdot\mathbf{B} = 0 \tag{1.3.S.3}$$

and so

$$\Phi = \int_{S'} da\,\hat{\mathbf{n}}\cdot\mathbf{B} = \int_{S} da\,\hat{\mathbf{n}}\cdot\mathbf{B} \tag{1.3.S.4}$$

Since it does not matter what surface S is used for computing the flux, we can then write,

$$\frac{d\Phi}{dt} = \frac{d}{dt}\left[\int_S da\,\hat{\mathbf{n}}\cdot\mathbf{B}\right] = \int_S da\,\hat{\mathbf{n}}\cdot\frac{\partial\mathbf{B}}{\partial t} \tag{1.3.S.5}$$

where S is held fixed, independent of time. This then leads to Eqs. (1.3.6) and (1.3.7) in the previous notes.

Motional emf

A more interesting case is where the loop C is not fixed, but may be moving in time. We can think of C as being a physical metallic wire loop containing conduction elections.

The emf is then defined in terms of the force $\mathbf{f}$ per unit charge that acts on charges traveling around the loop. If $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields acting on the loop, then $\mathbf{f} = \mathbf{E} + (\mathbf{v}/c)\times\mathbf{B}$, and the emf is defined by the integral around the loop,

$$\mathcal{E} = \oint_C \boldsymbol{d\ell}\cdot\mathbf{f} = \oint_C \boldsymbol{d\ell}\cdot\mathbf{E} + \oint_C \boldsymbol{d\ell}\cdot[(\mathbf{v}/c)\times\mathbf{B}] \tag{1.3.S.6}$$

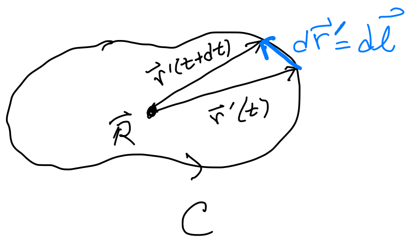
The addition of a possible magnetic force on the charges in the loop is a generalization of our expression in Eq. (1.3.S.1).

We first consider the case where the $\mathbf{E}$ and $\mathbf{B}$ fields are *constant* in time. We will then see that if the loop is moving, there will be an $\mathcal{E} \neq 0$ induced in the loop, but it will be due to the $\mathbf{B}$ part, rather than the $\mathbf{E}$ part of $\mathbf{f}$.

Since this is an electrostatic situation ($\mathbf{E}$ constant in time), we know that $\boldsymbol{\nabla} \times \mathbf{E} = 0$ everywhere, and so $\oint_C \boldsymbol{d\ell} \cdot \mathbf{E} = 0$. If there is to be an emf, it must come from $\mathbf{B}$.

Let us assume that the loop C has a rigid shape (it is possible to generalize to the case where the shape of the loop changes in time, but then the calculation is a bit messier – see Griffiths, 4th ed., pgs 307-308), and the loop is moving with a constant velocity $\mathbf{u}$ in a magnetic field $\mathbf{B}(\mathbf{r})$ that is constant in time, but can be spatially varying.

Let $\mathbf{R}$ denote the center of mass of the loop, and $\mathbf{r}'$ is the relative coordinate of a point on the loop with respect to $\mathbf{R}$. Then $\mathbf{r} = \mathbf{R} + \mathbf{r}'$ is the position of a charge as it goes around the loop. When we integrate around the loop to compute the emf, we then have for the differential tangent to the loop, $\boldsymbol{d\ell} = d\mathbf{r}'$.



The velocity of the charge is,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{r}'}{dt} = \mathbf{u} + \frac{d\mathbf{r}'}{dt} \tag{1.3.S.7}$$

and so the emf is,

$$\mathcal{E} = \frac{1}{c}\oint_C \boldsymbol{d\ell} \cdot (\mathbf{v} \times \mathbf{B}) = \frac{1}{c}\oint_C d\mathbf{r}' \cdot \left[\left(\mathbf{u} + \frac{d\mathbf{r}'}{dt}\right) \times \mathbf{B}\right] = \frac{1}{c}\oint_C d\mathbf{r}' \cdot \left(\frac{d\mathbf{r}'}{dt} \times \mathbf{B}\right) + \frac{1}{c}\oint_C d\mathbf{r}' \cdot (\mathbf{u} \times \mathbf{B}) \tag{1.3.S.8}$$

The first term can be written as,

$$\frac{1}{c}\oint_C d\mathbf{r}' \cdot \left(\frac{d\mathbf{r}'}{dt} \times \mathbf{B}\right) = \frac{1}{c}\oint_D dt\, \frac{d\mathbf{r}'}{dt} \cdot \left(\frac{d\mathbf{r}'}{dt} \times \mathbf{B}\right) = 0 \tag{1.3.S.9}$$

since $d\mathbf{r}'/dt$ must be orthogonal to $(d\mathbf{r}'/dt) \times \mathbf{B}$.

Thus the emf is due solely to the motion of the loop as a whole,

$$\mathcal{E} = \frac{1}{c}\oint_C d\mathbf{r}' \cdot (\mathbf{u} \times \mathbf{B}) \tag{1.3.S.10}$$

This will vanish if the loop is not moving, i.e. if $\mathbf{u} = 0$, but can be finite if the loop is moving.

We now rewrite the above using Stokes' theorem of vector calculus,

$$\mathcal{E} = \frac{1}{c}\oint_C d\mathbf{r}' \cdot (\mathbf{u} \times \mathbf{B}) = \frac{1}{c}\int_S d^2r'\, \hat{\mathbf{n}} \cdot [\boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B})] \tag{1.3.S.11}$$

where S is a surface bounded by C, and $\hat{\mathbf{n}}$ is its outward normal.

We can use the general vector identity

$$\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{a} - (\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{b} + \mathbf{a}(\boldsymbol{\nabla} \cdot \mathbf{b}) - \mathbf{b}(\boldsymbol{\nabla} \cdot \mathbf{a}) \tag{1.3.S.12}$$

and apply it with $\mathbf{b} = \mathbf{B}$ and $\mathbf{a} = \mathbf{u}$ a constant, to get,

$$\boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) = -(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{B} + \mathbf{u}(\boldsymbol{\nabla} \cdot \mathbf{B}) = -(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{B} \qquad \text{since } \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{1.3.S.13}$$

and so we then get,

$$\mathcal{E} = -\frac{1}{c} \int_S d^2r' \, \hat{\mathbf{n}} \cdot [(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{B}] \tag{1.3.S.14}$$

Finally we can write $(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{B} = \dfrac{d\mathbf{B}}{dt}$ as we show below:

As we integrate over the surface S bounded by the moving loop C, we are integrating over $\mathbf{B}(\mathbf{R}(t) + \mathbf{r}')$. We then can write,

$$\frac{d\mathbf{B}}{dt} = \sum_{i=1}^{3} \frac{\partial \mathbf{B}}{\partial r_i} \frac{dR_i}{dt} = \sum_{i=1}^{3} u_i \frac{\partial \mathbf{B}}{\partial r_i} = (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{B} \tag{1.3.S.15}$$

assuming that $\mathbf{B}(\mathbf{r})$ is itself not changing in time.

Since $d\mathbf{B}/dt$ is the total derivative of $\mathbf{B}$ with respect to time, which already takes into account that the loop and hence S is moving, we can take this derivative outside the integral and thus get,

$$\mathcal{E} = -\frac{1}{c} \int_S d^2r' \, \hat{\mathbf{n}} \cdot \frac{d\mathbf{B}}{dt} = -\frac{1}{c} \frac{d}{dt} \left[\int_S d^2r' \, \hat{\mathbf{n}} \cdot \mathbf{B} \right] = -\frac{1}{c} \frac{d\Phi}{dt} \tag{1.3.S.16}$$

We have thus arrived at Faraday's Law of induction, applied to *motinal emf*, i.e. when the fields are static but the loop is moving. In this case there is no electric field in the loop – the force driving the charges around the loop is the Lorentz force from the spatially non-uniform magnetic field acting on the moving loop.

Consider now the situation in the inertial frame of reference in which the above loop is stationary. In this case $\mathbf{u} = 0$ and so there is no contribution to the emf from the $\mathbf{B}$ field as above. However in this frame of reference, the magnetic field is no longer constant in time. If one sits at a fixed position $\mathbf{r}$ in this stationary *rest frame* of the loop, one will see a $\mathbf{B}$ that changes in time, since in this frame the spatially non-uniform magnetic field is moving with velocity $-\mathbf{u}$.

By the principle of relativity we expect that we must still find and emf given by $\mathcal{E} = -d\Phi/dt$ even in this rest frame of the loop (otherwise we would have a way to know the absolute state of motion of the loop).

Since the motional part of the emf due to $\mathbf{B}$ must be zero, since the loop is not moving, we then conclude that the emf must now be due to an $\mathbf{E}$ field in the loop, so that $\mathcal{E} = \oint_C \boldsymbol{d\ell} \cdot \mathbf{E} \neq 0$.

So, unlike electrostatics where $\oint_C \boldsymbol{d\ell} \cdot \mathbf{E} = 0$ always, once we have a time varying magnetic field we must have,

$$\mathcal{E} = \oint_C \boldsymbol{d\ell} \cdot \mathbf{E} = -\frac{1}{c} \frac{d\Phi}{dt} \qquad \Rightarrow \qquad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \tag{1.3.S.17}$$

and so we now recover Faraday's law as presented in the previous notes, and as first stated at the start of these notes.

We can thus conclude that Faraday's law is a consequence of special relativity combined with electro- and magnetostatics, and the Lorentz force. In the frame in which the loop moves with the constant velocity $\mathbf{u}$, there is a magnetic field but no electric field. The emf is due entirely to the magnetic field. In the rest frame of the loop, there is now still a magnetic field, but also an electric field. The emf is due entirely to the electric field. Thus, as we will see at the end of the course, when one makes a transformation from one inertial frame to another, $\mathbf{B}$ fields can turn into $\mathbf{E}$ fields, and vice versa.