Unit 1-5-S: Supplementary Material: Fourier Series to Fourier Transforms

Here I will review the derivation of the Fourier transform as a limit of the Fourier Series. I assume you have seen Fourier Series.

Fourier Series

Any function $f(x)$ defined on an interval $x \in [-\frac{L}{2}, \frac{L}{2}]$, and periodic on that interval, i.e., $f(L/2) = f(-L/2)$, can be expressed as a Fourier Series in terms of a basis set of cosine and sine functions,

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{2\pi n x}{L}\right) + B_n \sin\left(\frac{2\pi n x}{L}\right) \right] \tag{1.5.S1}$$

where

$$A_n = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x) \cos\left(\frac{2\pi n x}{L}\right) \qquad \text{and} \qquad B_n = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x) \sin\left(\frac{2\pi n x}{L}\right) \tag{1.5.S2}$$

To prove the above, we use the orthogonality of the cosine and sine functions. For m and n integers,

$$\frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \cos\left(\frac{2\pi m x}{L}\right) \cos\left(\frac{2\pi n x}{L}\right) = \delta_{mn} \tag{1.5.S3}$$

$$\frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \sin\left(\frac{2\pi m x}{L}\right) \sin\left(\frac{2\pi n x}{L}\right) = \delta_{mn} \tag{1.5.S4}$$

$$\frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \cos\left(\frac{2\pi m x}{L}\right) \sin\left(\frac{2\pi n x}{L}\right) = 0 \tag{1.5.S5}$$

You can derive these orthogonality integrals by rewriting the cosines and sines in terms of the complex exponential

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{1.5.S6}$$

substituting into the above, and then doing the integrals!

Accepting the orthogonality integrals of Eqs. (1.5.S3)–(1.5.S5), multiply Eq. (1.5.S1) by $\frac{L}{2}\cos(\frac{2\pi m x}{L})$ and integrate,

$$\frac{L}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \cos\left(\frac{2\pi m x}{L}\right) f(x) = \frac{L}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \cos\left(\frac{2\pi m x}{L}\right) \frac{A_0}{2} \tag{1.5.S7}$$

$$+ \sum_{n=1}^{\infty} \left[A_n \frac{L}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \cos\left(\frac{2\pi m x}{L}\right) \cos\left(\frac{2\pi n x}{L}\right) \right] \tag{1.5.S8}$$

$$+ \sum_{n=1}^{\infty} \left[B_n \frac{L}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \cos\left(\frac{2\pi m x}{L}\right) \sin\left(\frac{2\pi n x}{L}\right) \right] \tag{1.5.S9}$$

If $m = 0$, then the first term on the right hand side of the above integrates to give A_0, and the other two terms vanish by orthogonality. If $m \neq 0$, then the first and third terms on the right hand side of the above vanish by orthogonality, and the second term integrates to give A_m. We thus have demonstrated the Fourier coefficient formula Eq. (1.5.S2) for the A_n.

Similarly, if we multiply Eq. (1.5.S1) by $\frac{L}{2}\sin(\frac{2\pi mx}{L})$ and integrate, we will find the Fourier coefficient formula Eq. (1.5.S2) for the B_n.

Fourier Transform

To derive the Fourier transform from the Fourier series, substitute the expressions of Eq. (1.5.S6) for the cosine and sine in terms of the complex exponential, into Eqs. (1.5.S1) and (1.5.S2). One gets,

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{A_n}{2}+\frac{B_n}{2i}\right)\mathrm{e}^{i\left(\frac{2\pi nx}{L}\right)} + \left(\frac{A_n}{2}-\frac{B_n}{2i}\right)\mathrm{e}^{-i\left(\frac{2\pi nx}{L}\right)}\right] \tag{1.5.S10}$$

where

$$\frac{A_n}{2} \pm \frac{B_n}{2i} = \frac{A_n}{2} \mp \frac{iB_n}{2} = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\frac{1}{2}\left[\cos\left(\frac{2\pi nx}{L}\right) \mp i\sin\left(\frac{2\pi nx}{L}\right)\right] \tag{1.5.S11}$$

$$= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\,\mathrm{e}^{\mp i\left(\frac{2\pi nx}{L}\right)} \tag{1.5.S12}$$

Now define

$$f_n \equiv \frac{A_n}{2} + \frac{B_n}{2i} = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\,\mathrm{e}^{-i\left(\frac{2\pi nx}{L}\right)} \tag{1.5.S13}$$

and

$$f_{-n} \equiv \frac{A_n}{2} - \frac{B_n}{2i} = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\,\mathrm{e}^{-i\left(\frac{2\pi(-n)x}{L}\right)} \tag{1.5.S14}$$

and

$$f_0 \equiv \frac{A_0}{2} \tag{1.5.S15}$$

With these we can then write

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \mathrm{e}^{i\left(\frac{2\pi nx}{L}\right)} \qquad \text{where} \qquad f_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\,\mathrm{e}^{-i\left(\frac{2\pi nx}{L}\right)} \tag{1.5.S16}$$

This is the Fourier series written in complex exponential form.

Now define $k_n \equiv \dfrac{2\pi n}{L}$, with $\Delta k \equiv k_{n+1} - k_n = \dfrac{2\pi}{L}$ and substitute into Eq. (1.5.S16) to get,

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} L\, f_n\, \mathrm{e}^{ik_n x} \tag{1.5.S17}$$

Define

$$\tilde{f}(k_n) \equiv \frac{L}{2\pi} f_n = \frac{1}{2\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\,\mathrm{e}^{-ik_n x} \tag{1.5.S18}$$

So now we can write

$$f(x) = \sum_{n=-\infty}^{\infty} \Delta k\, \tilde{f}(k_n)\, \mathrm{e}^{ik_n x} \tag{1.5.S19}$$

Now take $L \to \infty$. In this limit, $\Delta k \to 0$, and $\sum \Delta k \to \int dk$. We thus get

$$\boxed{f(x) = \int_{-\infty}^{\infty} dk\, \tilde{f}(k)\, \mathrm{e}^{ikx} \qquad \text{and} \qquad \tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x)\, \mathrm{e}^{-ikx}} \tag{1.5.S20}$$

This is the Fourier transform and its inverse. We say that $\tilde{f}(k)$ is the Fourier transform of $f(x)$.