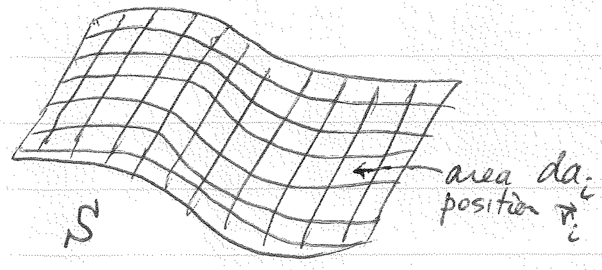
<u>Math Review</u> <u>Surface integrals</u> over surface S



tile surface with infinitesimally small tiles of area da_i at positions $\vec{r}_i$ on S

$$\int_S da\, f(\vec{r}) \equiv \sum_i f(\vec{r}_i)\, da_i$$

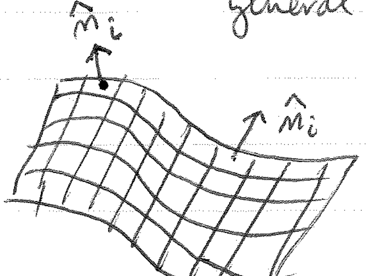
easy to evaluate analytically only for simple geometries.

<u>example</u> S = flat rectangular surface at $x \in [x_0, x_1]$, $y \in [y_0, y_1]$, $z = z_0$ constant.

$$\Rightarrow \int_S da\, f(\vec{r}) = \int_{x_0}^{x_1} dx \int_{y_0}^{y_1} dy\; f(x, y, z_0)$$

<u>vector surface integral</u> $\displaystyle\int_S d\vec{a} \cdot \vec{u}(\vec{r}) \equiv \int_S da\; \hat{n} \cdot \vec{u}(\vec{r})$

where $\hat{n}$ is the outward pointing unit vector normal to the surface S at point $\vec{r}$; direction of $\hat{n}$ will in general vary as position $\vec{r}$ on surface varies.

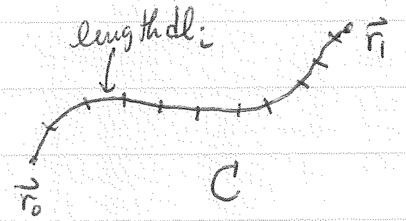


$$\int_S d\vec{a} \cdot \vec{u}(\vec{r}) = \sum_i da_i\; \hat{n}_i \cdot \vec{u}(\vec{r}_i)$$

for flat rectangle example above, $\hat{n} = \hat{z}$ everywhere on S

$$\int_S d\vec{a} \cdot \vec{u}(\vec{r}) = \int_{x_0}^{x_1} dx \int_{y_0}^{y_1} dy\;\; \hat{z} \cdot \vec{u}(x, y, z_0) = \int_{x_0}^{x_1} dx \int_{y_0}^{y_1} dy\;\; u_z(x, y, z_0)$$

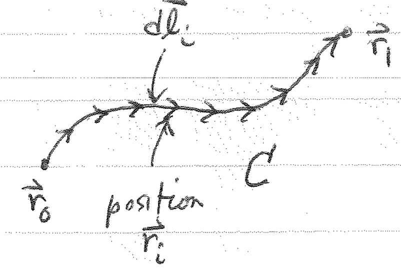
line integral along curve C



make curve C out of infinitesmally small straight line segments of length dl_i located at positions $\vec{r}_i$ on C

$$\int_C dl\, f(\vec{r}) \equiv \sum_i f(\vec{r}_i) dl_i$$

vector line integral



make curve out of infintesmal displacements $d\vec{l}_i$; direction of $d\vec{l}_i$ is tangent to C.

$$\int_C d\vec{l} \cdot \vec{u}(\vec{r}) \equiv \sum_i \vec{u}(\vec{r}_i) \cdot d\vec{l}_i$$

it is easy to convert vector line integral into an ordinary one dimensional integral, provided one has a parameterization of the curve C. For example, suppose curve is given by parameterization $\vec{r}(t) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z}$
where $\vec{r}(t_0) = \vec{r}_0$ and $\vec{r}(t_1) = \vec{r}_1$
are end points

Denote the points $\vec{r}_i = \vec{r}(t_i)$ where $t_i - t_{i-1} = \Delta t$

$$\int_C d\vec{l} \cdot \vec{u}(\vec{r}) = \sum_i \vec{u}(\vec{r}_i)\, d\vec{l}_i$$

By our definition, $d\vec{l}_i = \vec{r}_i - \vec{r}_{i-1} = \vec{r}(t_i) - \vec{r}(t_{i-1}) \simeq \frac{d\vec{r}}{dt}\Delta t$

$$\int_C d\vec{l} \cdot \vec{u}(\vec{r}) = \sum_i \vec{u}(\vec{r}(t_i)) \cdot \frac{d\vec{r}(t_i)}{dt} \Delta t$$

$$= \int_{t_0}^{t_1} dt \, \frac{d\vec{r}}{dt} \cdot \vec{u}(\vec{r}(t))$$

Physical example: let $\vec{r}(t)$ be the trajectory of a particle in time as it moves under the action of a force $\vec{F}$. The work done on the particle is

work: $$W = \int_C d\vec{l} \cdot \vec{F} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} \, dt$$

here $\frac{d\vec{r}}{dt}$ is just the velocity $\vec{v}$

$$\Rightarrow W = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{v}(t) \, dt$$

But $\vec{F} \cdot \vec{v}$ is just the power that force expends on moving particle

$$\Rightarrow W = \int_{t_0}^{t_1} (\text{power}) \, dt$$

So by parameterizing the particles trajectory in terms of time t, we recover the familiar result from mechanics that

work done = time integral of power

<u>another example</u> : suppose we have a curve C in xy-plane that we can write as $y = y(x)$, $z = z_0$

$\Rightarrow$ can parameterize the curve as

$$\vec{r}(x) = x\,\hat{x} + y(x)\,\hat{y} + z_0\,\hat{z} \qquad \text{from } x = x_0 \text{ to } x = x_1$$

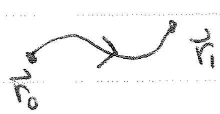
$$\int_C d\vec{l} \cdot \vec{u}(\vec{r}) = \int_{x_0}^{x_1} dx \, \frac{d\vec{r}}{dx} \cdot \vec{u}(\vec{r}(x))$$

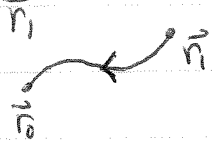
$$= \int_{x_0}^{x_1} dx \left[\hat{x} + \frac{dy}{dx}\,\hat{y} \right] \cdot \vec{u}(x, y(x), z_0)$$

$$= \int_{x_0}^{x_1} dx \left[u_x(x, y(x), z_0) + \frac{dy(x)}{dx}\, u_y(x, y(x), z_0) \right]$$

<u>Note</u>: direction of doing vector line integral is important

$$\int_{\vec{r}_0}^{\vec{r}_1} d\vec{l} \cdot \vec{u}(\vec{r}) = - \int_{\vec{r}_1}^{\vec{r}_0} d\vec{l} \cdot \vec{u}(\vec{r})$$



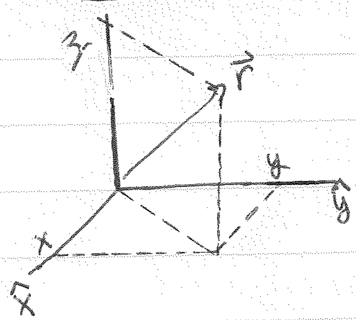


Math Review

Coordinate systems - orthonormal, right handed

1) "Cartesian" or "rectangular" coordinates

> in some texts, use $\hat{i}, \hat{j}, \hat{k}$ instead of $\hat{x}, \hat{y}, \hat{z}$



position vector

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

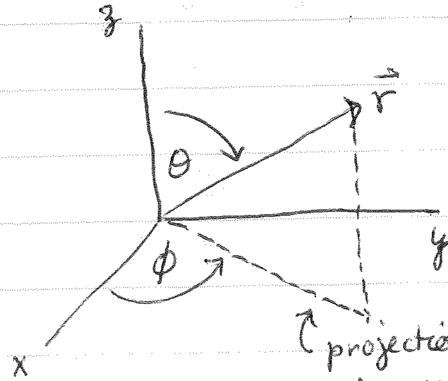
any vector

$$\vec{A}(\vec{r}) = A_x(\vec{r})\,\hat{x} + A_y(\vec{r})\,\hat{y} + A_z(\vec{r})\,\hat{z}$$

differential displacement : $d\vec{r}$

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

2) Spherical coordinates (r, θ, ϕ)



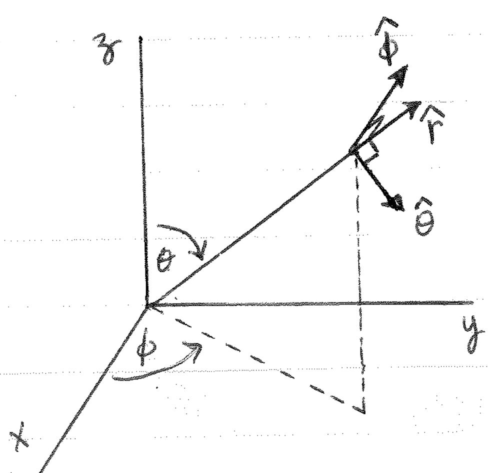
projection of $\vec{r}$ into xy plane

if $\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$, then

$$z = r\cos\theta$$
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$

$$r = \sqrt{x^2+y^2+z^2} = |\vec{r}|$$

unit basis vectors in spherical coords: $\hat{r}, \hat{\theta}, \hat{\phi}$



$\hat{\phi}$ lies in xy plane, ie $\hat{z}\cdot\hat{\phi} = 0$

position vector

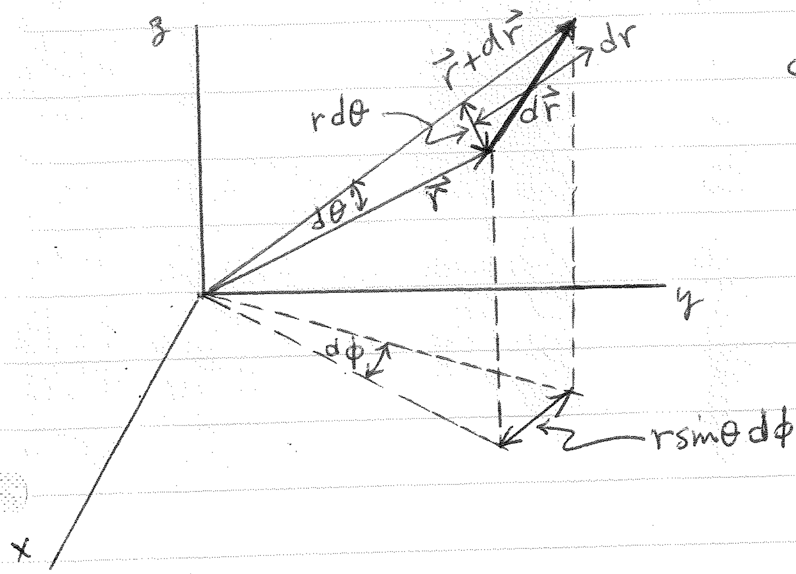
$$\vec{r} = r\,\hat{r}$$

any vector

$$\vec{A}(\vec{r}) = A_r(\vec{r})\,\hat{r} + A_\theta(\vec{r})\,\hat{\theta} + A_\phi(\vec{r})\,\hat{\phi}$$

key difference between spherical and Cartesian coords:
the directions of $\hat{x}, \hat{y}, \hat{z}$ is independent of position $\vec{r}$.
the directions of $\hat{r}, \hat{\theta}, \hat{\phi}$ vary as position $\vec{r}$ varies.

differential displacement: $d\vec{r}$



$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

since $\hat{r}, \hat{\theta}, \hat{\phi}$ are orthogonal, the volume swept out as r, θ, ϕ vary by $dr, d\theta, d\phi$ is:

differential volume element:

$$d^3r = (dr)(r\,d\theta)(r\sin\theta\,d\phi)$$
$$= dr\,d\theta\,d\phi\; r^2\sin\theta$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, f(x,y,z) = \int_0^{\infty} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\; r^2 \sin\theta\; f(r,\theta,\phi)$$

differential surface element – for surface at fixed radius $r = R$

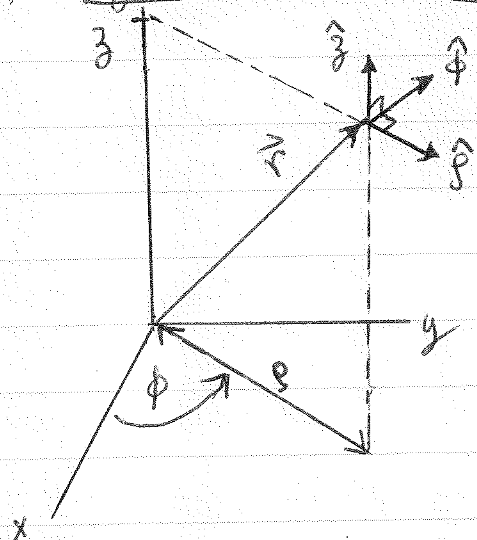
$$da = R^2 \sin\theta\, d\theta\, d\phi$$

$$\int_S da\, f(\vec{r}) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\; R^2\sin\theta\; f(R,\theta,\phi)$$

example: surface of sphere of radius R: use above with $f = 1$

$$\text{Area} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, R^2 \sin\theta = 2\pi R^2 (-\cos\theta)\Big|_0^{\pi} = 4\pi R^2$$

3) <u>Cylindrical coordinates</u> (ρ, ϕ, z)



$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

$$\Rightarrow \quad x = \rho\cos\phi$$
$$y = \rho\sin\phi$$

$$\rho = \sqrt{x^2+y^2}$$

unit basis vectors in cylindrical coords: $\hat{\rho}, \hat{\phi}, \hat{z}$

both $\hat{\rho}$ and $\hat{\phi}$ lie in xy plane. directions of $\hat{\rho}, \hat{\phi}$ depend on position $\vec{r}$

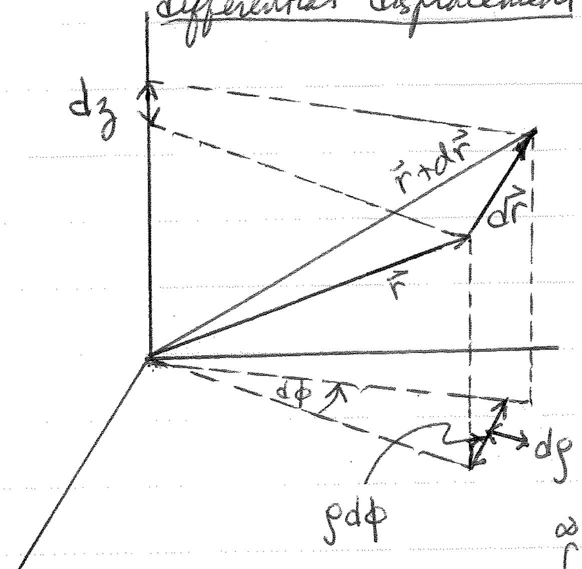
position vector

$$\vec{r} = \rho\,\hat{\rho} + z\,\hat{z}$$

any vector

$$\vec{A}(\vec{r}) = A_\rho(\vec{r})\,\hat{\rho} + A_\phi(\vec{r})\,\hat{\phi} + A_z(\vec{r})\,\hat{z}$$

<u>differential displacement</u> $d\vec{r}$



$$d\vec{r} = d\rho\,\hat{\rho} + \rho\, d\phi\,\hat{\phi} + dz\,\hat{z}$$

<u>differential volume element:</u>

$$d^3r = (d\rho)(\rho\, d\phi)(dz)$$
$$= d\rho\, d\phi\, dz\, \rho$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\; f(x,y,z) = \int_0^{\infty} d\rho \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\; \rho\, f(\rho,\phi,z)$$

<u>differential surface element</u> - for surface of cylinder at fixed radius R, and length from $z = L_0$ to $z = L_1$

$$da = R\, d\phi\, dz$$

$$\int_S da\, f(\vec{r}) = \int_{L_0}^{L_1} dz \int_0^{2\pi} d\phi \;\; R\, f(R, \phi, z)$$

Review of vector differential operators (§ 1-2, 1-3)

Gradient: (1.2.2)

$$f(\vec{r}+d\vec{r}) = f(x+dx,\ y+dy,\ z+dz)$$
$$= f(x,y,z) + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{df}{dz}dz$$

define gradient vector

$$\vec{\nabla}f(x,y,z) \equiv \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

$$\Rightarrow f(\vec{r}+d\vec{r}) = f(\vec{r}) + (\vec{\nabla}f)\cdot d\vec{r} \qquad \text{since } d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

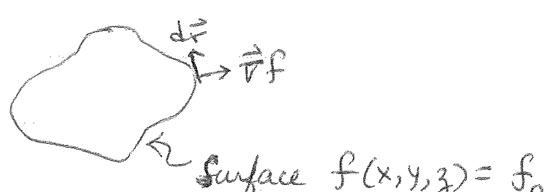
$$df \equiv f(\vec{r}+d\vec{r}) - f(\vec{r}) = (\vec{\nabla}f)\cdot(d\vec{r}) = |\vec{\nabla}f|\,|d\vec{r}|\cos\theta$$

where θ is angle between $\vec{\nabla}f$ and $d\vec{r}$

$\Rightarrow$ geometrical meanings of gradient:

1) df is max, for a given $|d\vec{r}|$, when $\theta = 0$, ie when $d\vec{r}$ points along $\vec{\nabla}f$. $\Rightarrow$ $\vec{\nabla}f$ points in direction of greatest increase in function f. $|\vec{\nabla}f|$ is the slope of f in this direction

2) $df = 0$ when $\theta = \frac{\pi}{2}$, ie. when $d\vec{r}$ is $\perp$ to $\vec{\nabla}f$.
$\Rightarrow$ $\vec{\nabla}f$ is normal to the surfaces of constant f.



3) $df = 0$ for all directions of $d\vec{r}$, if $\vec{\nabla}f = 0$.
$\Rightarrow$ $\vec{\nabla}f(\vec{r}_0) = 0$ means $\vec{r}_0$ is a max, min, or saddle point of $f(\vec{r})$

Differential operator $\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$

gradient $f = \vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$

<u>Divergence</u> (1.2.4)

apply $\vec{\nabla}$ to a vector function $\vec{v}(\vec{r})$

$$\vec{\nabla}\cdot\vec{v} \equiv \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}v_x + \hat{y}v_y + \hat{z}v_z\right)$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \equiv \text{divergence of } \vec{v}$$

<u>Curl or circulation</u> (1.2.5)

$$\vec{\nabla}\times\vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\times\left(\hat{x}v_x + \hat{y}v_y + \hat{z}v_z\right)$$

$$\cancel{= \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)}$$

$$\cancel{= \left(\frac{\partial}{\partial x}v_y - \frac{\partial}{\partial y}v_x\right)\hat{z}}$$

$$\equiv \left(\frac{\partial}{\partial y}v_z - \frac{\partial}{\partial z}v_y\right)\hat{x} + \left(\frac{\partial}{\partial z}v_x - \frac{\partial}{\partial x}v_z\right)\hat{y}$$

$$+ \left(\frac{\partial}{\partial x}v_y - \frac{\partial}{\partial y}v_x\right)\hat{z}$$

To remember how to compute curl, just think of cyclic permutations of x y z

x y z	$(\vec{A}\times\vec{B})_x$	$= A_y B_z - A_z B_y$
y z x	$(\vec{A}\times\vec{B})_y$	$= A_z B_x - A_x B_z$
z x y	$(\vec{A}\times\vec{B})_z$	$= A_x B_y - A_y B_x$

use rules of differentiation to get product rules like:

$$\vec{\nabla}\cdot(f\vec{A}) = \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z)$$

f ← scalar, $\vec{A}$ ← vector

$$= f\frac{\partial A_x}{\partial x} + A_x\frac{\partial f}{\partial x} + f\frac{\partial A_y}{\partial y} + A_y\frac{\partial f}{\partial y}$$

$$+ f\frac{\partial A_z}{\partial z} + A_z\frac{\partial f}{\partial z}$$

$$\Rightarrow \quad \vec{\nabla}\cdot(f\vec{A}) = f\,\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}f$$

or

$$\vec{\nabla}\times(f\vec{A}) = \left[\frac{\partial}{\partial y}(fA_z) - \frac{\partial}{\partial z}(fA_y)\right]\hat{x} + \left[\frac{\partial}{\partial z}(fA_x) - \frac{\partial}{\partial x}(fA_z)\right]\hat{y}$$

$$+ \left[\frac{\partial}{\partial x}(fA_y) - \frac{\partial}{\partial y}(fA_x)\right]\hat{z}$$

$$= \left[f\frac{\partial A_z}{\partial y} - f\frac{\partial A_y}{\partial z} + \frac{\partial f}{\partial y}A_z - \frac{\partial f}{\partial z}A_y\right]\hat{x}$$

$$+ \left[f\frac{\partial A_x}{\partial z} - f\frac{\partial A_z}{\partial x} + \frac{\partial f}{\partial z}A_x - \frac{\partial f}{\partial x}A_z\right]\hat{y}$$

$$+ \left[f\frac{\partial A_y}{\partial x} - f\frac{\partial A_x}{\partial y} + \frac{\partial f}{\partial x}A_y - \frac{\partial f}{\partial y}A_x\right]\hat{z}$$

$$\Rightarrow \quad \vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) + (\vec{\nabla}f)\times\vec{A}$$

order of terms is important – it is $(\vec{\nabla}f)\times\vec{A}$ and not $\vec{A}\times(\vec{\nabla}f)$

see sec 1.2.6 for other examples

Second derivatives (1.2.7)

1) Laplacian

$$\vec{\nabla}\cdot(\vec{\nabla}f) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \equiv \nabla^2 f$$

2) $$\vec{\nabla}\times(\vec{\nabla}f) = \left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right]\hat{x}$$

$$+\left[\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)\right]\hat{y}$$

$$+\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right]\hat{z}$$

$$= 0 \quad \text{since } \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} \quad \text{etc.}$$

$\vec{\nabla}\times(\vec{\nabla}f) = 0$ for <u>any</u> scalar function $f(\vec{r})$

3) $$\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) = \hat{x}\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) + \hat{y}\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

$$+ \hat{z}\frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

$$= \hat{x}\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial x \partial y} + \frac{\partial^2 v_z}{\partial x \partial z}\right) + \text{etc.}$$

4) $$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$= \frac{\partial^2 v_z}{\partial x \partial y} - \frac{\partial^2 v_y}{\partial x \partial z} + \frac{\partial^2 v_x}{\partial y \partial z} - \frac{\partial^2 v_z}{\partial y \partial x} + \frac{\partial^2 v_y}{\partial z \partial x} - \frac{\partial^2 v_x}{\partial z \partial y} = 0.$$

So $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = 0$ for any vector function $\vec{v}$

5) $$\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \hat{x}\left[\frac{\partial}{\partial y}(\vec{\nabla}\times\vec{v})_z - \frac{\partial}{\partial z}(\vec{\nabla}\times\vec{v})_y\right] + \hat{y}\left[\frac{\partial}{\partial z}(\vec{\nabla}\times\vec{v})_x - \frac{\partial}{\partial x}(\vec{\nabla}\times\vec{v})_z\right] + \hat{z}\left[\frac{\partial}{\partial x}(\vec{\nabla}\times\vec{v})_y - \frac{\partial}{\partial y}(\vec{\nabla}\times\vec{v})_x\right]$$

$$= \hat{x}\left[\frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\right] + \hat{y}\left[\frac{\partial}{\partial z}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right] + \hat{z}\left[\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\right]$$

$$= \hat{x}\left[-\frac{\partial^2 v_x}{\partial y^2} - \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial^2 v_y}{\partial x\partial y} + \frac{\partial^2 v_z}{\partial x\partial z}\right] + \hat{y}\left[-\frac{\partial^2 v_y}{\partial x^2} - \frac{\partial^2 v_y}{\partial z^2} + \frac{\partial^2 v_x}{\partial y\partial x} + \frac{\partial^2 v_z}{\partial y\partial z}\right] + \hat{z}\left[-\frac{\partial^2 v_z}{\partial x^2} - \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_x}{\partial z\partial x} + \frac{\partial^2 v_y}{\partial z\partial y}\right]$$

$$= \hat{x}\left[-\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \frac{\partial^2 v_x}{\partial x\partial x} + \frac{\partial^2 v_y}{\partial x\partial y} + \frac{\partial^2 v_z}{\partial x\partial z}\right] + \hat{y}\left[-\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \frac{\partial^2 v_x}{\partial y\partial x} + \frac{\partial^2 v_y}{\partial y\partial y} + \frac{\partial^2 v_z}{\partial y\partial z}\right] + \hat{z}\left[-\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \frac{\partial^2 v_x}{\partial z\partial x} + \frac{\partial^2 v_y}{\partial z\partial y} + \frac{\partial^2 v_z}{\partial z\partial z}\right]$$

$$= \hat{x}\left[-\nabla^2 v_x\right] + \hat{y}\left[-\nabla^2 v_y\right] + \hat{z}\left[-\nabla^2 v_z\right] + \vec{\nabla}(\vec{\nabla}\cdot\vec{v})$$

(see (3))

Define Laplacian of vector function:

$$\nabla^2 \vec{v} \equiv \hat{x}(\nabla^2 v_x) + \hat{y}(\nabla^2 v_y) + \hat{z}(\nabla^2 v_z)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = -\nabla^2 \vec{v} + \vec{\nabla}(\vec{\nabla} \cdot \vec{v})$$

It is only easy to write what we mean by $\nabla^2 \vec{v}$ in Cartesian coordinates

or we could define more generally

$$\nabla^2 \vec{v} \equiv \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{v})$$

could represent these in spherical or cylindrical coords (coming soon!) but result does not look simple

<u>Example</u> $\vec{E}(\vec{r}) = \dfrac{Q}{r^2}\hat{r}$ electric field from point charge Q at origin

What is $\vec{\nabla}\cdot\vec{E}$?

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

need to express $\vec{E}$ in Cartesian coordinates to find E_x, E_y, E_z

$$\vec{E} = Q\frac{\hat{r}}{r^2} = Q\frac{\vec{r}}{r^3} = Q\frac{(x\hat{x} + y\hat{y} + z\hat{z})}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial E_x}{\partial x} = Q\frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= Q\left[\frac{1}{r^3} - \frac{3x^2}{r^5}\right]$$

Similarly, $\dfrac{\partial E_y}{\partial y} = \left(\dfrac{1}{r^3} - \dfrac{3y^2}{r^5}\right)Q$ and $\dfrac{\partial E_z}{\partial z} = \left(\dfrac{1}{r^3} - \dfrac{3z^2}{r^5}\right)Q$

So

$$\vec{\nabla}\cdot\vec{E} = Q\left[\frac{1}{r^3} - \frac{3x^2}{r^5} + \frac{1}{r^3} - \frac{3y^2}{r^5} + \frac{1}{r^3} - \frac{3z^2}{r^5}\right]$$

$$= Q\left[\frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{3}{r^2} - \frac{3r^2}{r^5}\right]$$

$$= 0!$$

So it looks like $\vec{\nabla}\cdot\vec{E} = 0$ for a point charge. This turns out to <u>not</u> be quite correct as we will soon see.

Define Laplacian of vector function

$$\nabla^2 \vec{v} \equiv \hat{x}(\nabla^2 v_x) + \hat{y}(\nabla^2 v_y) + \hat{z}(\nabla^2 v_z)$$

$$\Rightarrow \quad \vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = -\nabla^2 \vec{v} + \vec{\nabla}(\vec{\nabla}\cdot\vec{v})$$

---

gradient, divergence, curl, Laplacian, in spherical + cylindrical coordinates

spherical coords: scaler func $f(r,\theta,\phi)$

gradient $\vec{\nabla} f$

you might think that $\vec{\nabla} f$ should be

$$\frac{\partial f}{\partial r}\hat{r} + \frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial \phi}\hat{\phi}$$

using analogy with the form in cartesian coords

$$\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}.$$

This is not correct. To get correct form, use definition:

$$df = f(\vec{r}+d\vec{r}) - f(\vec{r}) \equiv (\vec{\nabla} f)\cdot d\vec{r}$$

in spherical coords:

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi \qquad (*)$$

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$df = (\vec{\nabla} f)\cdot d\vec{r}$$

$$\Rightarrow \quad df = (\vec{\nabla} f)_r\,dr + (\vec{\nabla} f)_\theta\,r\,d\theta + (\vec{\nabla} f)_\phi\,r\sin\theta\,d\phi$$

compare with $(*)$ $\Rightarrow$ $(\vec{\nabla} f)_r = \frac{\partial f}{\partial r}$, $(\vec{\nabla} f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$, $(\vec{\nabla} f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$

$$\boxed{\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}}$$

$$\Rightarrow \quad \vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

<u>divergence</u>

$$\vec{\nabla}\cdot\vec{v} = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\cdot\left(\hat{r}v_r + \hat{\theta}v_\theta + \hat{\phi}v_\phi\right)$$

you might think this should be:

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

But this is <u>not</u> correct - because when we look at any particular term, for example $\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\cdot(\hat{r}v_r)$ we need to remember that $\hat{r}, \hat{\theta}, \hat{\phi}$ vary with position, + so derivatives of basis vectors must also be taken into account

$$\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\cdot(\hat{r}v_r) = \underbrace{\hat{\theta}\cdot\hat{r}}_{=0}\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{v_r}{r}\underbrace{\hat{\theta}\cdot\frac{\partial\hat{r}}{\partial\theta}}_{\neq 0}$$

when take this into account correctly, find

$$\boxed{\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}}$$

* Note common mistake: $\vec{\nabla}\cdot\vec{v} \neq \frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_\phi}{\partial\phi}$!!

Similarly for curl:

$$\boxed{\begin{aligned}\vec{\nabla}\times\vec{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} \\ &+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}\end{aligned}}$$

Laplacian:

$$\boxed{\begin{aligned}\nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) \\ &+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\end{aligned}}$$

* Note common mistake: $\nabla^2 f \neq \frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial\phi^2}$!!!

One can similarly find for cylindrical coordinates:

gradient: $\vec{\nabla} f = \frac{\partial f}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial\phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$

divergence $\vec{\nabla}\cdot\vec{v} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho v_\rho) + \frac{1}{\rho}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$

curl: $\vec{\nabla}\times\vec{v} = \left[\frac{1}{\rho}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\rho} + \left[\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial\rho}\right]\hat{\phi} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho v_\phi) - \frac{\partial v_\rho}{\partial\phi}\right]\hat{z}$

Laplacian $\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}$

Example ① $\vec{E}(\vec{r}) = Q \dfrac{\hat{r}}{r^2}$ electric field from point charge Q at origin

Compute $\vec{\nabla}\cdot\vec{E}$ in spherical coords!

Since $E_\theta = E_\phi = 0$ and $E_r = Q \dfrac{1}{r^2}$

we have

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \, Q \, \frac{1}{r^2}\right)$$

$$= Q \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0 \, !$$

Same answer as before! But maybe $\neq 0$ at $r=0$!

Example ② Consider electric field from sphere of radius R filled with uniform charge density ρ_0

$\Rightarrow$ total charge on sphere is $Q = \frac{4}{3}\pi R^3 \rho_0$

electric field is $\vec{E}(\vec{r}) = \begin{cases} Q \dfrac{\hat{r}}{r^2} & r > R \text{ outside} \\ Q \dfrac{r\,\hat{r}}{R^3} & r < R \text{ inside} \end{cases}$

so $\vec{\nabla}\cdot\vec{E} = 0$ for $r > R$ outside, as in Example ①

for $r < R$,

$$\vec{\nabla}\cdot\vec{E} = \frac{Q}{R^3} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot r\right)$$

$$= \frac{Q}{R^3}\frac{1}{r^2} 3r^2 = \frac{3Q}{R^3} = 4\pi\rho_0$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho_0$$

So for uniformly charged sphere,

$$\vec{\nabla}\cdot\vec{E} = \begin{cases} 0 & r > R \quad \text{outside} \\ 4\pi\rho_0 & r < R \quad \text{inside} \end{cases}$$

Now suppose we viewed the point charge of $\vec{E} \times \textcircled{1}$ not as a true point charge, but rather as a sphere of very small radius R with charge density $\rho_0 = \dfrac{Q}{\frac{4}{3}\pi R^3}$

We would then conclude that $\vec{\nabla}\cdot\vec{E}$ for the pt charge was

$$\vec{\nabla}\cdot\vec{E} = \begin{cases} 0 & r > R \quad \text{outside} \\ \dfrac{3Q}{R^3} & r < R \quad \text{inside} \end{cases}$$

As $R \to 0$, approaching the limit of a true point charge, then $\vec{\nabla}\cdot\vec{E} = 0$ ~~almost~~ everywhere except at $r=0$ where $\vec{\nabla}\cdot\vec{E} = \dfrac{3Q}{R^3}$ is growing

infinitely large! as $R \to 0$.

The function $\vec{\nabla}\cdot\vec{E}$ above thus has peculiar property that it is zero everywhere except at a single point, at which it is infinite; moreover the integral of $\vec{\nabla}\cdot\vec{E}$ over all space just gives $4\pi Q$. We will soon see that a

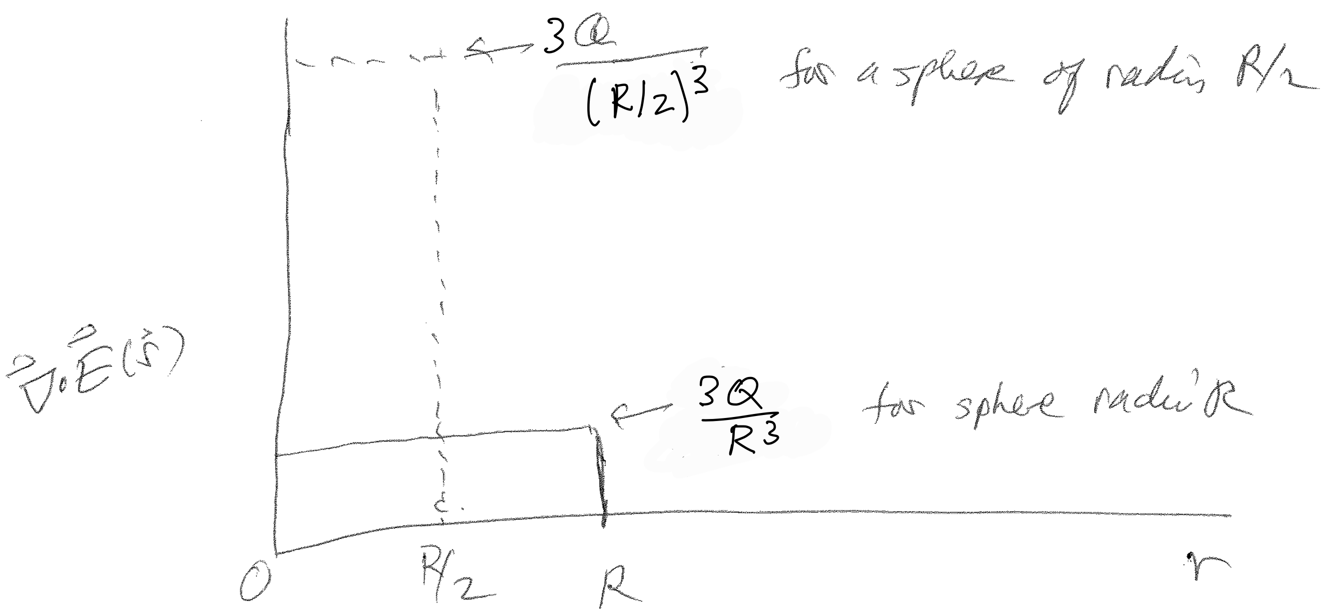
function with this strange behavior is called the Dirac delta function

Note: When we directly computed for the point charge

$$\vec{\nabla}\cdot\vec{E} = Q \frac{1}{r^2}\frac{\partial}{\partial r}(1)$$

this really gives zero only when $r > 0$.
When $r = 0$, the above is really an indeterminant expression of the form $\frac{0}{0}$
so we don't know what is its real value

The preceeding example tells us that the value of $\vec{\nabla}\cdot\vec{E}$ at $r=0$ is really infinite.



as radius of sphere decrease, height of $\vec{\nabla}\cdot\vec{E}$ goes up. As $R \to 0$, $\vec{\nabla}\cdot\vec{E} \to \infty$ at $r=0$

Integral vector calculus (Griffiths §1-3)

1) Gradients + line Integrals

$$\boxed{\int_{\vec{r}_a}^{\vec{r}_b} d\vec{l} \cdot \vec{\nabla} f = f(\vec{r}_b) - f(\vec{r}_a)}$$

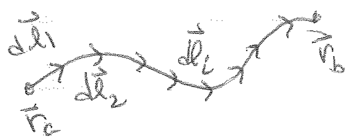
along path C

independent of path C

proof: for a pt on curve $\vec{r}$,

$$d\vec{l} \cdot \vec{\nabla} f = f(\vec{r} + d\vec{l}) - f(\vec{r})$$

by definition of gradient

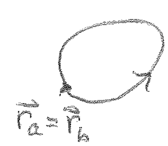


$$\int d\vec{l} \cdot \vec{\nabla} f = \left[f(\vec{r}_a + d\vec{l}_1) - f(\vec{r}_a)\right] + \left[f(\vec{r}_a + d\vec{l}_1 + d\vec{l}_2) - f(\vec{r}_a + d\vec{l}_1)\right] + \cdots \left[f(\vec{r}_b) - f(\vec{r}_b - d\vec{l}_N)\right]$$

$$= f(\vec{r}_b) - f(\vec{r}_a)$$

$$\Rightarrow \oint_C d\vec{l} \cdot \vec{\nabla} f = 0$$

if C is a closed path



as then $\vec{r}_a = \vec{r}_b$

start = end

2) Divergences + Surface Integrals : Gauss's Theorem

$$\boxed{\int_V d^3r \, (\vec{\nabla} \cdot \vec{v}) = \oint_S \vec{v} \cdot d\vec{a}}$$

where S is closed surface bounding volume V.

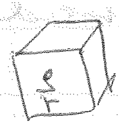
$d\vec{a} = \hat{n}\, da$ where $\hat{n}$ is outward normal to S.

Gauss's theorem provides the geometrical meaning for divergence operator.

$\oint_S \vec{v} \cdot d\vec{a}$ is "flux of v" through the surface S

If $\vec{v}$ represents velocity field of a fluid, then $\oint \vec{v} \cdot d\vec{a}$ gives the total rate that fluid is flowing out the surface S. (see prob 1.32)

divergence $\vec{\nabla} \cdot \vec{v}(\vec{r})$ then gives the flux of $\vec{v}$ out of the pt $\vec{r}$. To see this:



ΔV small, so that $\vec{v}$ is $\approx$ constant over volume ΔV

$$\Rightarrow \int_{\Delta V} d^3r\, \vec{\nabla} \cdot \vec{v} \cong \Delta V \left[\vec{\nabla} \cdot \vec{v}(r)\right] = \oint_S \vec{v} \cdot d\vec{a}$$

$$\vec{\nabla} \cdot \vec{v}(\vec{r}) \simeq \frac{1}{\Delta V} \oint_S \vec{v} \cdot d\vec{a}$$

= flux per unit volume of $\vec{v}$ out of the point S.

3) ~~Curl and line integrals : Stokes Theorem~~

$$\cancel{\int_S d\vec{a} \cdot (\vec{\nabla} \times \vec{v}) = \oint_C \vec{v} \cdot d\vec{l}}$$

$\Rightarrow \oint_S \vec{v} \cdot d\vec{a} = 0$ unless $\vec{\nabla} \cdot \vec{v} \neq 0$ somewhere inside S

$$\oint_{S_1} \vec{v} \cdot d\vec{a} = \oint_{S_2} \vec{v} \cdot d\vec{a}$$ provided $\vec{\nabla} \cdot \vec{v} = 0$ in region between S_1 and S_2

<u>Example:</u> Consider $\vec{E}(\vec{r}) = Q \dfrac{\hat{r}}{r^2}$ for a point charge at origin. Compute $\oint_S \vec{E} \cdot d\vec{a}$ for the surface S of a sphere of radius R. Since $d\vec{a} = da\,\hat{n} = da\,\hat{r}$ for $\vec{r}$ on S,

$$\vec{E}(\vec{r}) \cdot d\vec{a} = Q \frac{\hat{r}}{R^2} \cdot da\,\hat{r} = Q \frac{1}{R^2}\, da$$

is <u>constant</u> over surface of sphere S

$$\Rightarrow \oint \vec{E}(\vec{r}) \cdot d\vec{a} = \left(Q \frac{1}{R^2} \right) \underbrace{\left(4\pi R^2 \right)}_{\text{area of } S}$$

flux of $\vec{E}$ through S $= 4\pi Q$ independent of radius R!

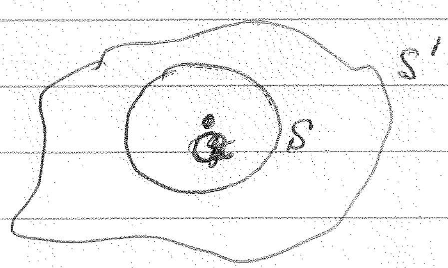
divergence of $\vec{E}$ at origin is therefore

$$\vec{\nabla} \cdot \vec{E}(0) = \frac{1}{\frac{4}{3}\pi R^3} \oint_S E(\vec{r}) \cdot d\vec{a} = \frac{3Q}{R^3} \to \infty$$

as $R \to 0$

This is the same answer we got from our model of the point charge as a uniformly charged sphere of small but finite radius.

Note that it did not matter what was the shape of the surface on which we did the integration



$$\frac{Q}{\varepsilon_0} = \oint_{S} d\vec{a}\cdot\vec{E}(\vec{r}) = \oint_{S'} d\vec{a}\cdot\vec{E}(\vec{r})$$

where the last equality follows since $\vec{\nabla}\cdot\vec{E} = 0$ everywhere inbetween S and S'

So we have

$$\oint_{S} d\vec{a}\cdot\vec{E} = \begin{cases} 4\pi Q & \text{for } \underline{\text{any}} \text{ surface } S \text{ that contains the charge } Q \\ 0 & \text{for any surface } S \text{ that does not contain } Q \end{cases}$$

equivalently

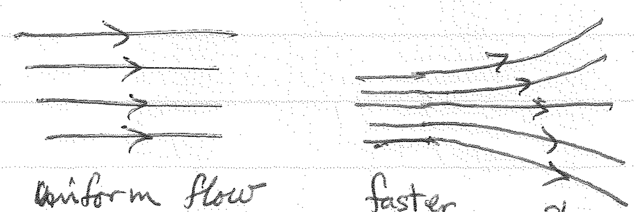
$$\vec{\nabla}\cdot\vec{E} = \begin{cases} 0 & \text{for all } \vec{r} \neq 0 \\ \dfrac{4\pi Q}{\Delta V} \to \infty & \text{at } r = 0, \text{ where } \Delta V \text{ is volume of small region containing } Q. \end{cases}$$

$\nabla\cdot\vec{E}$ therefore has the following properties: it vanishes <u>everywhere</u> except at $\vec{r}=0$. At $\vec{r}=0$ it is infinite in just the right way that $\int d^3r\, \vec{\nabla}\cdot\vec{E}$ is a finite constant $4\pi Q$. We will see that a function which has this behavior is the "Dirac delta function" "field lines"

draw lines pointed in direction of $\vec{v}(\vec{r})$

density of lines proportional to $|\vec{v}(\vec{r})|$

if $\vec{v}$ is velocity field:



uniform flow

faster flow

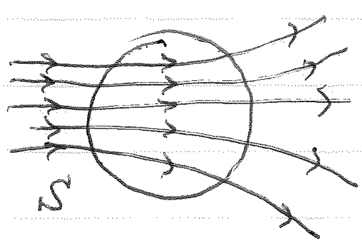
slower flow

$\int_S d\vec{a}\cdot\vec{v} \propto$ number of lines passing through S counting with (+) sign if line goes out and with (−) sign if line goes in

If field lines are continuous, then $\vec{\nabla}\cdot\vec{v} = 0$

$\vec{\nabla}\cdot\vec{v} \neq 0$ only at point where field lines become singular.

example

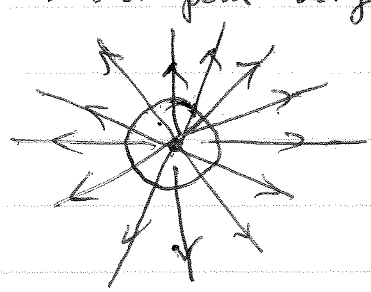


$\oint_S d\vec{a}\cdot\vec{v} = 0$ since field lines continuous, the number of lines going into S = number of lines going out of S. This will be true for any surface S.

$$0 = \oint_S d\vec{a}\cdot\vec{v} = \int_V d^3r\ \vec{\nabla}\cdot\vec{v} \quad \text{for any } S$$

$$\Rightarrow \vec{\nabla}\cdot\vec{v} = 0$$

example: $\vec{E}$ from point charge



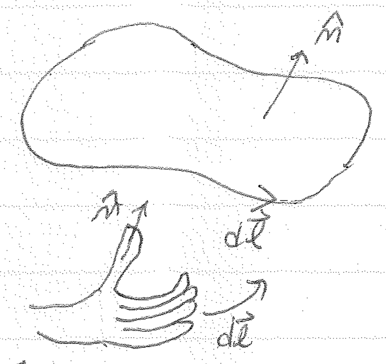
$\oint_S d\vec{a}\cdot\vec{E} > 0$ as lines only pass out of S.

$\Rightarrow \vec{\nabla}\cdot\vec{E} \neq 0$ somewhere inside of S. (at center where charge is)

3) Curl and Line Integrals : Stokes theorem

$$\int_S d\vec{a}\cdot(\vec{\nabla}\times\vec{v}) = \oint_C d\vec{l}\cdot\vec{v}$$

where S is an open surface with boundary C



Right Hand!

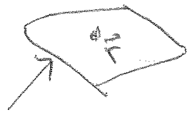
must choose $d\vec{a} = \hat{n}\,da$ and $d\vec{l}$ consistent with right hand rule, ie. if allign right thumb along $\hat{n}$ (normal to surface) then $d\vec{l}$ must be in direction that fingers point along

geometrical meaning for curl operator

$\oint_C d\vec{l}\cdot\vec{v}$ is the "circulation" of $\vec{v}$ around the loop C

If $\vec{v}$ represents velocity of fluid, and C is a pipe containing fluid (or $\vec{v}$ is vellocity of electrons and C is wire loop) then $\oint_C d\vec{l}\cdot\vec{v}$ gives the net circulation of fluid going around in the loop.

$\vec{\nabla}\times\vec{v}$ gives the circulation of $\vec{v}$ at the pt $\vec{r}$



ΔA

$$\int_{\Delta A} d\vec{a}\cdot(\vec{\nabla}\times\vec{v}) \cong \Delta A\,\hat{n}\cdot(\vec{\nabla}\times\vec{v}(\vec{r})) = \oint d\vec{l}\cdot\vec{v}$$

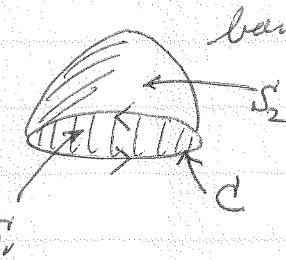
normal to ΔA

$$\hat{n}\cdot\left[\vec{\nabla}\times\vec{v}(\vec{r})\right] = \frac{1}{\Delta A}\oint d\vec{l}\cdot\vec{v}$$ circulation per unit area of $\vec{v}$ at point $\vec{r}$.

$\Rightarrow \int_S d\vec{a} \cdot (\vec{\nabla} \times \vec{v})$ depends only on the boundary line of S

$$\int_{S_1} d\vec{a} \cdot (\vec{\nabla} \times \vec{v}) = \int_{S_2} d\vec{a} \cdot (\vec{\nabla} \times \vec{v})$$

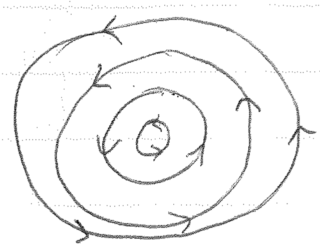
if S_1 and S_2 have same boundary C



$\Rightarrow \oint_S d\vec{a} \cdot \vec{\nabla} \times \vec{v} = 0$ for closed surface S, as boundary curve $C = 0$!

in <u>terms of field lines:</u>

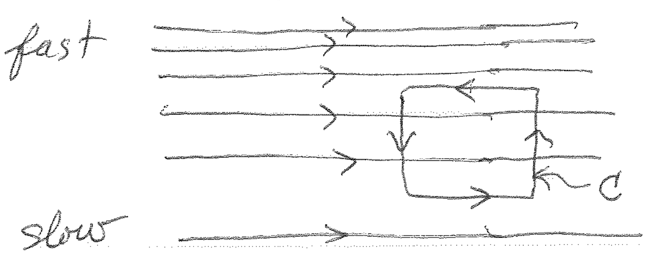
example



Clearly, if we have field lines that close upon themselves, then we must have $\vec{\nabla} \times \vec{v} \neq 0$ somewhere inside the loop as $\oint d\vec{l} \cdot \vec{v} \neq 0$ along such a curve

But we can also have $\vec{\nabla} \times \vec{v} \neq 0$ in other situations:

Shear flow:



$\oint_C d\vec{l} \cdot \vec{v} \neq 0$!

Dirac δ-function (§ 1-5)

we saw $\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 0$ everywhere except at $\vec{r}=0$

But $\oint_S d\vec{a}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi$ for any S that encloses $\vec{r}=0$

$$\| $$

$$\int_{V} d^3r\ \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) \Rightarrow \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) \to \infty \text{ at } \vec{r}=0,$$

V enclosed by S $\Rightarrow \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)$ is not an ordinary continuous function

This motivates the Dirac δ-function

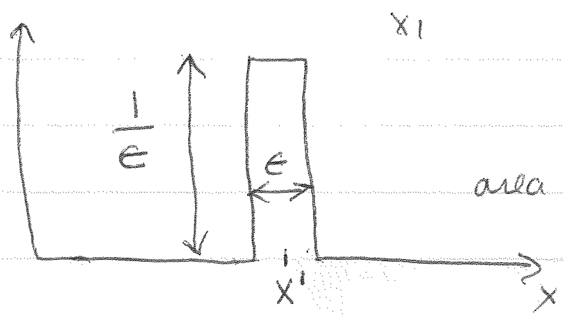
$$\delta(x-x') = \begin{cases} 0 & x \neq x' \\ \infty & x = x' \end{cases}$$

and
$$\int_{x_1}^{x_2} dx\ \delta(x-x') = \begin{cases} 1 & \text{if } x_1 < x' < x_2 \\ 0 & \text{otherwise} \end{cases}$$

Can think of $\delta(x-x')$ as being a limit of a sequence of functions:

Let
$$f_\epsilon(x-x') = \begin{cases} 0 & \text{if } |x-x'| > \frac{\epsilon}{2} \\ \frac{1}{\epsilon} & \text{if } |x-x'| < \frac{\epsilon}{2} \end{cases}$$

$$\int_{x_1}^{x_2} dx\ f(x-x') = \begin{cases} 1 & \text{if } x_1 < x'-\frac{\epsilon}{2} \text{ and } x_2 > x'+\frac{\epsilon}{2} \\ 0 & \text{if } x_2 < x'-\frac{\epsilon}{2} \text{ or } x_1 > x'+\frac{\epsilon}{2} \end{cases}$$



area under curve = 1

$$\delta(x-x') = \lim_{\epsilon\to 0} f_\epsilon(x-x')$$

Properties of $\delta(x-x')$

$$\int_{x_1}^{x_2} dx \; g(x)\,\delta(x-x') = \begin{cases} g(x') & \text{if } x_1 < x' < x_2 \\ 0 & \text{otherwise} \end{cases}$$

Proof: since integrand is zero everywhere except at $x=x'$ we can evaluate $g(x)$ at x' and make it

$$= \int_{x_1}^{x_2} dx \; g(x')\,\delta(x-x') = g(x') \underbrace{\int_{x_1}^{x_2} dx \; \delta(x-x')}_{= \begin{cases} 1 & \text{if } x_1 < x' < x_2 \\ 0 & \text{otherwise} \end{cases}}$$

Ex: Consider

$$\int_{-\infty}^{\infty} dx \; g(x)\,\delta(ax+b) = \,?$$

let $y \equiv ax+b \qquad dx = \frac{dy}{a}$

if $a>0$

$$= \int_{-\infty}^{\infty} \frac{dy}{a}\, g\left(\frac{y-b}{a}\right) \delta(y) = \frac{g\left(\frac{-b}{a}\right)}{a}$$

if $a<0$

$$= \int_{+\infty}^{-\infty} \frac{dy}{a}\, g\left(\frac{y-b}{a}\right) \delta(y) = -\int_{-\infty}^{\infty} \frac{dy}{a}\, g\left(\frac{y-b}{a}\right) \delta(y)$$

$$= -\frac{g\left(\frac{-b}{a}\right)}{a}$$

general $\Rightarrow$ $$\int_{-\infty}^{\infty} dx \; g(x)\,\delta(ax+b) = \frac{g\left(-\frac{b}{a}\right)}{|a|}$$

$$\Rightarrow \boxed{\delta(ax+b) = \frac{1}{|a|}\,\delta\left(x+\frac{b}{a}\right)}$$

because $\int dx\, g(x) \dfrac{\delta(x+\frac{b}{a})}{|a|} = \dfrac{1}{|a|} g(-\frac{b}{a})$

In general, if $D_1(x)$ and $D_2(x)$ are two expressions involving δ-functions, then $D_1 = D_2$ if

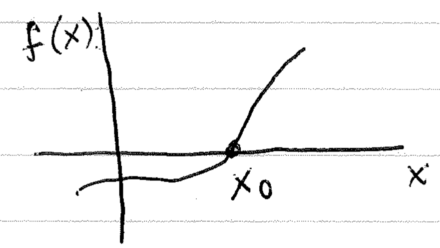
$$\int dx\, g(x)\, D_1(x) = \int dx\, g(x) D_2(x)$$

for any function $g(x)$

Another property of the Dirac δ-function

$$\int_{x_1}^{x_2} dx\, g(x)\, \delta(f(x)) = \frac{g(x_0)}{\left|\frac{df(x_0)}{dx}\right|}$$

if f is monotonic increasing or decreasing with x_0 such that $f(x_0)=0$ and $x_1 < x_0 < x_2$



To see this, note that the only place the integrand is non-zero is when the argument of the δ-function vanishes, ie when $f(x)=0$. This happens at $x=x_0$. So we can expand $f(x)$ in Taylor series about x_0. To lowest order we have

$$f(x) \simeq f(x_0) + \left(\frac{df(x_0)}{dx}\right)(x-x_0) = \frac{df(x_0)}{dx}(x-x_0)$$

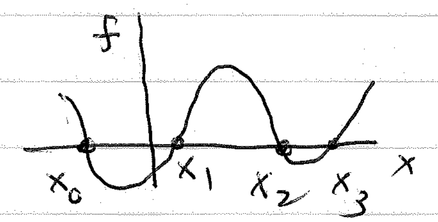
since $f(x_0)=0$. $f(x)$ now has the form

$f(x) = ax+b$ with $a = \dfrac{df(x_0)}{dx}$ and $b = -\left(\dfrac{df}{dx}(x_0)\right)x_0$

So from previous example we get

$$\int dx \; g(x)\, \delta(f(x)) = \frac{g(x_0)}{\left|\frac{df(x_0)}{dx}\right|}$$

For a more general $f(x)$ that is not monotonic and may have several zeros at $x_0, x_1, x_2, \ldots$ we have



$$\int_{x_a}^{x_b} dx \; g(x)\, \delta(f(x)) = \sum_{\substack{i \\ \text{such that} \\ x_a < x_i < x_b}} \frac{g(x_i)}{\left|\frac{df(x_i)}{dx}\right|}$$

3-dimensional δ-function

$$\delta^3(\vec{r}-\vec{r}\,') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

$$\int_V f(\vec{r})\, \delta^3(\vec{r}-\vec{r}\,')\, d^3r = \begin{cases} f(\vec{r}\,') & \text{if } \vec{r}\,' \in V \\ 0 & \text{if } \vec{r}\,' \notin V \end{cases}$$

Recall we said

$$\boxed{\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \nabla\cdot\left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right) = 4\pi\,\delta^3(\vec{r}-\vec{r}\,')}$$

Very important result to remember!!

since $\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 0$ except at $\vec{\mathscr{r}} = \vec{r}-\vec{r}\,' = 0$

and $\displaystyle\int_V d^3r \; \vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \oint_S d\vec{a}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \begin{cases} 4\pi & \text{if } V \text{ contains } \vec{r}\,' \\ 0 & \text{otherwise} \end{cases}$

Now we saw in workshop

$$\vec{\nabla}\left(\frac{1}{r}\right) = \frac{-\hat{r}}{r^2}$$

In workshop we did this calculation in Cartesian coords.

Now to see this we can do the differentiation in spherical coordinates

$$\vec{\nabla}\left(\frac{1}{r}\right) = \hat{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \frac{\hat{\theta}}{r}\underbrace{\frac{\partial}{\partial\theta}\left(\frac{1}{r}\right)}_{=0} + \frac{\hat{\varphi}}{r\sin\theta}\underbrace{\frac{\partial}{\partial\varphi}\left(\frac{1}{r}\right)}_{=0}$$

$$= \frac{-\hat{r}}{r^2}$$

So

$$\vec{\nabla}\cdot\left(\vec{\nabla}\left(\frac{1}{r}\right)\right) = \vec{\nabla}\cdot\left(\frac{-\hat{r}}{r^2}\right) = -4\pi\delta^3(\vec{r})$$

or

$$\boxed{\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r})}$$

very important to remember!