Unit 1-2: Magnetsostatics

The theory of magnetostatics was developed over the years by doing experiments on the forces produced by wires carrying a steady electric current. Here we will not follow that historical development but instead use an alternative presentation that yields the same results.

Just like a stationary electric charge produces an electric field, an electric charge in motion is found to produce a magnetic field. For a charge q', located at the orgin $\mathbf{r}' = 0$, and moving with a velocity $\mathbf{v}'$ (where $|\mathbf{v}'| \ll c$, with c the speed of light in the vacuum) a magnetic field will be produced at position $\mathbf{r}$ equal to

$$\mathbf{B}(\mathbf{r}) = k_2 q' \mathbf{v}' \times \frac{\mathbf{r}}{r^3} = \frac{k_2}{k_1} \mathbf{v}' \times \mathbf{E}(\mathbf{r}). \tag{1.2.1}$$

Here $\mathbf{E}(\mathbf{r}) = k_1 q' \mathbf{r}/r^3$ is the electrostatic field produced by q', and k_2 is a new universal constant of nature. Just as we saw that the value of k_1 is set by the units we choose for charge, so here the value of k_2 is set by the units we choose for magnetic field.

Lorentz Force

A magnetic field exerts a force on a charge that is moving. The total electric and magnetic force acting on a charge is called the Lorentz force. For a charge q at position $\mathbf{r}$ moving with velocity $\mathbf{v}$, the Lorentz force q feels due to the electric and magnetic fields at its position is

$$\mathbf{F} = q \left[\mathbf{E}(\mathbf{r}) + k_3 \mathbf{v} \times \mathbf{B}(\mathbf{r})\right] \tag{1.2.2}$$

The first term is the familiar Coulomb force, the second term is the magnetic force. The constant k_3 is another universal constant of nature (by *universal* we mean it has the same value for all charges no matter where they are or how they are moving). However, unlike k_1 and k_2, we cannot choose this new constant k_3 to have any value we like – defining the units of charge q has already been used to set the value of k_1, while defining the units of magnetic field $\mathbf{B}$ has been used to set the value of k_2. There is no further freedom of units to play with, so given choices for k_1 and k_2, the value of k_3 is fixed by nature.

Combining the above, the force on a charge q at position $\mathbf{r}$, moving with velocity $\mathbf{v}$, due to a charge q' at the origin $\mathbf{r}' = 0$, moving with velocity $\mathbf{v}'$ is, in the non-relativistic limit $|\mathbf{v}|, |\mathbf{v}'| \ll c$,

$$\mathbf{F} = k_1 q q' \frac{\mathbf{r}}{r^3} + k_2 k_3 q q' \frac{\mathbf{v} \times (\mathbf{v}' \times \mathbf{r})}{r^3} \tag{1.2.3}$$

The first term is just the Coulomb force, the second term is the magnetic analog of the Coulomb force.

The magnetic part is just the point charge equivalent of the Biot-Savart Law for the force between current carrying wires. If we regard $q\mathbf{v} = \mathbf{I}$ as the current of charge q, and $q'\mathbf{v}' = \mathbf{I}'$ as the current of charge q', then the magnetic force is $k_2 k_3 \mathbf{I} \times (\mathbf{I}' \times \mathbf{r})/r^3$, which is just the Biot-Savart Law.

We can rewrite the above force as

$$\mathbf{F} = k_1 q q' \left(1 + \frac{k_2 k_3}{k_1} \mathbf{v} \times \mathbf{v}' \times \right) \frac{\mathbf{r}}{r^3} \tag{1.2.4}$$

From this we see that $k_2 k_3/k_1$ must have the units of (velocity)$^{-2}$. Since velocity has units that are unaffected by whatever we choose for the units of charge or magnetic field, then the numerical value of the combination $k_2 k_3/k_1$ must be independent of whatever choices we made for our units of q and $\mathbf{B}$, and so be independent of the unit-dependent values of k_1 and k_2. Experimentally it is found that

$$\frac{k_2 k_3}{k_1} = \frac{1}{c^2} \qquad \text{where } c \text{ is the speed of light in the vacuum.} \tag{1.2.5}$$

Continuum Current Density

For charges q_i at positions $\mathbf{r}_i(t)$, with velocities $\mathbf{v}_i = d\mathbf{r}_i/dt$, we define the continuum current density as,

$$\mathbf{j}(\mathbf{r},t) = \sum_i q_i\,\mathbf{v}_i(t)\,\delta(\mathbf{r}-\mathbf{r}_i(t)) \tag{1.2.6}$$

This expression treats each charge i as if it gives a current $q_i\mathbf{v}_i$ located at the position $\mathbf{r}_i$ of the charge.

Noting that the units of the delta function are $1/(\text{length})^3$, the units of $\mathbf{j}$ are

$$(\text{charge})\left(\frac{\text{length}}{\text{time}}\right)\left(\frac{1}{\text{length}^3}\right) = \left(\frac{\text{charge}}{\text{area}\cdot\text{time}}\right) = \left(\frac{\text{current}}{\text{area}}\right) \tag{1.2.7}$$

For a surface S

$$\int_S da\,\hat{\mathbf{n}}\cdot\mathbf{j} = I \qquad \text{the total current (charge per unit time) flowing through the surface } S. \tag{1.2.8}$$

Charge Conservation

For a volume V bounded by a surface S, charge conservation can be written as,

$$\begin{array}{ccc} \dfrac{d}{dt}\left[\displaystyle\int_V d^3r\,\rho(\mathbf{r},t)\right] & = & -\displaystyle\oint_S da\,\hat{\mathbf{n}}\cdot\mathbf{j}(\mathbf{r},t) \\ \begin{array}{c}\text{rate of change of total}\\ \text{charge in } V\end{array} & = & \begin{array}{c}\text{(-) rate of charge}\\ \text{flowing out of } V\\ \text{through the surface } S\end{array} \end{array} \tag{1.2.9}$$

Using Gauss' Theorem we can then write

$$\oint_S da\,\hat{\mathbf{n}}\cdot\mathbf{j} = \int_V d^3r\,\boldsymbol{\nabla}\cdot\mathbf{j} = -\int_V d^3r\,\frac{\partial\rho(\mathbf{r},t)}{\partial t} \tag{1.2.10}$$

Since this must hold true for any volume V, the integrands of the last two terms must be equal. So this results in the law of local charge conservation,

$$\boxed{\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0} \tag{1.2.11}$$

A static situation has a charge density that does not change in time, so $\partial\rho/\partial t = 0$. From this, and the above conservation law, we see that the condition for a magnetostatic situation is $\boldsymbol{\nabla}\cdot\mathbf{j} = 0$. For a magnetostatic situation, charges may be moving, but they must move in such a way that whatever charge leaves a point in space, it is immediately replaced by an equal amount of charge flowing into that point in space, so that ρ stays the same. The condition that requires outflow = inflow is that the current density is divergenceless.

Maxwell's Equations for Magnetostatics

For a set of charges q_i at positions $\mathbf{r}_i$, moving with velocities $\mathbf{v}_i$, we can generalize Eq. (1.2.1) to,

$$\mathbf{B}(\mathbf{r}) = \sum_i k_2\,q_i\,\mathbf{v}_i\times\frac{(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|^3} = k_2\int d^3r'\,\mathbf{j}(\mathbf{r}')\times\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = k_2\int d^3r'\,\mathbf{j}(\mathbf{r}')\times\boldsymbol{\nabla}\left(\frac{-1}{|\mathbf{r}-\mathbf{r}'|}\right) \tag{1.2.12}$$

To check that the middle step is correct, just substitute in our definition of the current density $\mathbf{j}$ from Eq. (1.2.6) and integrate over the delta functions. The last step follows from Eq. (1.1.15).

For any constant vector $\mathbf{A}$ and any scalar function $\phi(\mathbf{r})$, we have the following vector identity: $\boldsymbol{\nabla}\times(\mathbf{A}\phi) = -\mathbf{A}\times\boldsymbol{\nabla}\phi$. Regarding $\mathbf{j}(\mathbf{r}')$ as $\mathbf{A}$, and $1/|\mathbf{r}-\mathbf{r}'|$ as ϕ, we can then use this to write,

$$\mathbf{B}(\mathbf{r}) = k_2\boldsymbol{\nabla}\times\left[\int d^3r'\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] \qquad \text{so } \mathbf{B} \text{ is the curl of some vector function.} \tag{1.2.13}$$

Note that although $\mathbf{j}(\mathbf{r}')$ varies with position $\mathbf{r}'$, it *is* a constant with respect to the operator $\boldsymbol{\nabla}$, since $\boldsymbol{\nabla}$ differentiates only with respect to the variable $\mathbf{r}$ and *not* with respect to the variable $\mathbf{r}'$.

From Eq. (1.2.13) it follows that,

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{B}(\mathbf{r}) = 0} \quad \text{since the divergence of the curl of any vector function must always vanish, } \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r})) = 0 \text{ for any } \mathbf{A}(\mathbf{r}). \tag{1.2.14}$$

The integral version of Eq. (1.2.14) is obtained from using Gauss' Theorem,

$$\int_V d^3r\,\boldsymbol{\nabla}\cdot\mathbf{B} = \boxed{\oint_S da\,\hat{\mathbf{n}}\cdot\mathbf{B} = 0} \tag{1.2.15}$$

Having found the divergence of $\mathbf{B}$, we now want to find its curl. Taking the curl of Eq. (1.2.13) we get,

$$\boldsymbol{\nabla}\times\mathbf{B}(\mathbf{r}) = k_2\boldsymbol{\nabla}\times\left[\boldsymbol{\nabla}\times\left(\int d^3r'\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)\right] \tag{1.2.16}$$

For any vector field $\mathbf{A}(\mathbf{r})$ we have the vector identity $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}$. Applying this identity to Eq. (1.2.16) gives,

$$\boldsymbol{\nabla}\times\mathbf{B}(\mathbf{r}) = k_2\boldsymbol{\nabla}\left[\int d^3r'\,\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)\right] - k_2\int d^3r'\,\nabla^2\left(\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) \tag{1.2.17}$$

and, since for a constant vector $\mathbf{A}$ and scalar function $\phi(\mathbf{r})$ one has $\nabla^2(\mathbf{A}\phi) = \mathbf{A}\nabla^2\phi$, we can write,

$$\boldsymbol{\nabla}\times\mathbf{B}(\mathbf{r}) = k_2\boldsymbol{\nabla}\left[\int d^3r'\,\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)\right] - k_2\int d^3r'\,\mathbf{j}(\mathbf{r}')\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \tag{1.2.18}$$

where we used the fact that $\mathbf{j}(\mathbf{r}')$ is just a constant vector with respect the ∇^2, since ∇^2 operates on $\mathbf{r}$ and not on $\mathbf{r}'$.

In the second term above we can use Eq. (1.1.30) to write $\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -4\pi\delta(\mathbf{r}-\mathbf{r}')$.

In the first term we can write, $\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) = \mathbf{j}(\mathbf{r}')\cdot\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -\mathbf{j}(\mathbf{r}')\cdot\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)$,

where we used that (i) for any constant vector $\mathbf{A}$ and any scalar function $\phi(\mathbf{r})$, $\boldsymbol{\nabla}\cdot(\mathbf{A}\phi) = \mathbf{A}\cdot\boldsymbol{\nabla}\phi$, and (ii) that, when applied to $1/|\mathbf{r}-\mathbf{r}'|$, $\boldsymbol{\nabla}' = -\boldsymbol{\nabla}$ (the first differentiates with respect to $\mathbf{r}'$, while the second differentiates with respect to $\mathbf{r}$).

Using these in Eq. (1.2.18) gives

$$\boldsymbol{\nabla}\times\mathbf{B}(\mathbf{r}) = -k_2\boldsymbol{\nabla}\left[\int d^3r'\,\mathbf{j}(\mathbf{r}')\cdot\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right] + 4\pi k_2\int d^3r'\,\mathbf{j}(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') \tag{1.2.19}$$

Integrating over the delta function in the second term just gives $4\pi k_2\,\mathbf{j}(\mathbf{r})$. To rewrite the first term we use the vector identity that for any vector function $\mathbf{A}(\mathbf{r})$ and any scalar function $\phi(\mathbf{r})$, we have $\boldsymbol{\nabla}\cdot(\mathbf{A}\phi) = \mathbf{A}\cdot\boldsymbol{\nabla}\phi + (\boldsymbol{\nabla}\cdot\mathbf{A})\phi$, which we can rearranged to be $\mathbf{A}\cdot\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}\cdot(\mathbf{A}\phi) - (\boldsymbol{\nabla}\cdot\mathbf{A})\phi$. Taking $\mathbf{j}$ as $\mathbf{A}$ and $1/|\mathbf{r}-\mathbf{r}'|$ as ϕ, and regarding these as functions of $\mathbf{r}'$, we can rewrite the integral in the first term of Eq. (1.2.19) as,

$$\int d^3r'\,\mathbf{j}(\mathbf{r}')\cdot\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = \int d^3r'\,\boldsymbol{\nabla}'\cdot\left(\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) - \int d^3r'\,\left[\boldsymbol{\nabla}'\cdot\mathbf{j}(\mathbf{r}')\right]\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{1.2.20}$$

For a magnetostatic situation, the second term above must vanish, since a magnetostatic situation is defined by the condition $\boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r}) = 0$. For the first term we can use Gauss' Theorem to write,

$$\int d^3r'\,\boldsymbol{\nabla}'\cdot\left(\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) = \int_S da'\,\hat{\mathbf{n}}\cdot\left(\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) \tag{1.2.21}$$

If we let the volume of integration be all of space, then the surface S enclosing that volume moves off to infinity, and if the current $\mathbf{j}$ is sufficiently localized so that $\mathbf{j}(\mathbf{r}) \to 0$ sufficiently rapidly as $|\mathbf{r}| \to \infty$, then the surface integral above will vanish. Thus we conclude that the first term in Eq. (1.2.19) vanishes, and we are left with,

$$\boxed{\boldsymbol{\nabla} \times \mathbf{B}(\mathbf{r}) = 4\pi k_2\, \mathbf{j}(\mathbf{r}) \qquad \text{Ampere's Law for magnetostatics}} \tag{1.2.22}$$

We can use Stokes Theorem to rewrite this in integral form,

$$\int_S da\, \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \mathbf{B} = \boxed{\oint_C \boldsymbol{d\ell} \cdot \mathbf{B} = 4\pi k_2 \int_S da\, \hat{\mathbf{n}} \cdot \mathbf{j} = 4\pi k_2 I_{\text{through}}} \tag{1.2.23}$$

where C is the curve bounding the surface S, $\boldsymbol{d\ell}$ is the differential tangent to the curve, and I_{through} is the current flowing through the surface.

We thus have Maxwell's equations for magnetostatics,

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{B}(\mathbf{r}) = 0, \qquad \boldsymbol{\nabla} \times \mathbf{B}(\mathbf{r}) = 4\pi k_2\, \mathbf{j}(\mathbf{r})} \tag{1.2.24}$$

Note, although we derived the above equations starting from a non-relativistic point charge version of the Biot-Savart Law, they turn out to remain correct for all magnetostatic situations.