Unit 1-4: Electromagnetic Potentials and Gauge Invariance

Maxwell's equations are,

$$1)\quad \boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\,\rho \qquad\qquad 3)\quad \boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{1.4.1}$$

$$2)\quad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \qquad\qquad 4)\quad \boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \tag{1.4.2}$$

We now want to show how to rewrite these equations using scalar and vector potentials. To start, we review the use of potentials in statics.

Electrostatics

In statics we have $\boldsymbol{\nabla}\times\mathbf{E} = 0$ since $\partial\mathbf{B}/\partial t = 0$. We know from vector calculus that if the curl of a vector function is everywhere zero, then we can write that vector function as the gradient of a scalar function ϕ.

$$\mathbf{E} = -\boldsymbol{\nabla}\phi \quad\Rightarrow\quad \boldsymbol{\nabla}\times\mathbf{E} = -\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = 0 \tag{1.4.3}$$

ϕ is called the electrostatic potential. In terms of this potential, Gauss' Law for electric fields become

$$\boldsymbol{\nabla}\cdot\mathbf{E} = -\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi) = -\nabla^2\phi = 4\pi\rho \tag{1.4.4}$$

so

$$\boxed{\nabla^2\phi = -4\pi\rho \qquad \text{Poisson's equation}} \tag{1.4.5}$$

In regions where $\rho = 0$ we have

$$\boxed{\nabla^2\phi = 0 \qquad \text{Laplace's equation}} \tag{1.4.6}$$

In our discussion of Coulomb's Law for electrostatics we saw that the electric field from a distribution of localized charge could be written as,

$$\mathbf{E}(\mathbf{r}) = \int d^3r'\,\rho(\mathbf{r}')\,\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\boldsymbol{\nabla}\left[\int d^3r'\,\frac{\rho(\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right] = -\boldsymbol{\nabla}\phi \tag{1.4.7}$$

We therefore see that the solution to Poisson's equation for a localized charge distribution ρ, with $\mathbf{E}\to 0$ as $|\mathbf{r}|\to\infty$, is

$$\phi(\mathbf{r}) = \int d^3r'\,\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{1.4.8}$$

We will soon spend a fair amount of time learning new ways to solve Poisson's equation, both for arbitrary ρ where we can approximate the above integral (this gives the multipole expansion), and for cases where ϕ or $\boldsymbol{\nabla}\phi$ are specified on the surfaces of specified regions of space, such as conducting surfaces (this will define boundary value problems).

Note, the electrostatic potential is not unique. One can always add to it an arbitrary constant $\phi' = \phi + C$, and if $\mathbf{E} = -\boldsymbol{\nabla}\phi$ then we will also have $\mathbf{E} = -\boldsymbol{\nabla}\phi'$. The potential becomes unique if we also add a boundary condition to the problem, such as $\phi(\mathbf{r})\to 0$ as $|\mathbf{r}|\to\infty$.

Magnetostatics

We have $\boldsymbol{\nabla}\cdot\mathbf{B}=0$. This is true generally, even if one is not in a magnetostatic situation. We know from vector calculus that if the divergence of a vector function vanishes everywhere, then it can always be written as the curl of another vector function $\mathbf{A}$.

$$\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A} \qquad \Rightarrow \qquad \boldsymbol{\nabla}\cdot\mathbf{B}=\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A})=0 \tag{1.4.9}$$

$\mathbf{A}$ is the magnetic vector potential.

Ampere's Law in magnetostatics is

$$\boldsymbol{\nabla}\times\mathbf{B}=\frac{4\pi}{c}\mathbf{j} \qquad \text{(in magnetostatics, } \partial\mathbf{E}/\partial t=0) \tag{1.4.10}$$

so expressed in terms of $\mathbf{A}$ it becomes

$$\boldsymbol{\nabla}\times\mathbf{B}=\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})-\nabla^2\mathbf{A}=\frac{4\pi}{c}\mathbf{j} \tag{1.4.11}$$

Magnetostatic gauge invariance

There are many possible vector potentials $\mathbf{A}$ that result in the same $\mathbf{B}$. If $\mathbf{A}$ is such that $\boldsymbol{\nabla}\times\mathbf{A}=\mathbf{B}$, then for any scalar function $\chi(\mathbf{r})$, we have that $\mathbf{A}'=\mathbf{A}+\boldsymbol{\nabla}\chi$ also gives $\boldsymbol{\nabla}\times\mathbf{A}'=\mathbf{B}$. This follows since $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\chi=0$ for any scalar $\chi(\mathbf{r})$. This freedom to change $\mathbf{A}$, without changing $\mathbf{B}$, by adding a term $\boldsymbol{\nabla}\chi$ is called *gauge invariance.*

Therefore we can always choose to represent $\mathbf{B}$ by a particular vector potential $\mathbf{A}$ they obeys some additional convenient condition. In magnetostatics one usually chooses $\mathbf{A}$ to satisfy $\boldsymbol{\nabla}\cdot\mathbf{A}=0$. In this case, since $\boldsymbol{\nabla}\cdot\mathbf{A}=0$, Ampere's Law in Eq. (1.4.11) simplifies to,

$$\boxed{\nabla^2\mathbf{A}=-\frac{4\pi}{c}\mathbf{j} \qquad \text{Poisson's equation}} \tag{1.4.12}$$

the same form as Eq. (1.4.5) as for the electrostatic potential ϕ. Therefore the solution to $\mathbf{A}$ from a localized current density has the same form as given by Eq. (1.4.8),

$$\mathbf{A}(\mathbf{r})=\frac{1}{c}\int d^3r'\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{1.4.13}$$

One could ask, how do we know it is always possible to find an $\mathbf{A}$ that satisfies both $\boldsymbol{\nabla}\times\mathbf{A}=\mathbf{B}$ *and* $\boldsymbol{\nabla}\cdot\mathbf{A}=0$?

proof: Suppose we had $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$ for some $\mathbf{A}$, but $\boldsymbol{\nabla}\cdot\mathbf{A}=D(\mathbf{r})\neq 0$. We could then construct an $\mathbf{A}'=\mathbf{A}+\boldsymbol{\nabla}\chi$ with χ chosen so as to give $\boldsymbol{\nabla}\cdot\mathbf{A}'=0$. To see this,

$$\text{If} \quad \boldsymbol{\nabla}\cdot\mathbf{A}'=\boldsymbol{\nabla}\cdot\mathbf{A}+\nabla^2\chi=0 \quad \text{then} \quad \nabla^2\chi=-\boldsymbol{\nabla}\cdot\mathbf{A}=-D \tag{1.4.14}$$

So the χ we want is just the solution to Poisson's equation where the source is $D(\mathbf{r})$.

$$\chi(\mathbf{r})=\frac{1}{4\pi}\int d^3r'\,\frac{D(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{1.4.15}$$

With this χ we construct $\mathbf{A}'$, and we will then have both $\boldsymbol{\nabla}\times\mathbf{A}'=\mathbf{B}$ and $\boldsymbol{\nabla}\cdot\mathbf{A}'=0$.

Back to Dynamics and the Full Maxwell Equations

In general we have $\boldsymbol{\nabla}\cdot\mathbf{B}=0$, so $\boxed{\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}}$ continues to remain true.

But now instead of $\boldsymbol{\nabla}\times\mathbf{E}=0$ we have $\boldsymbol{\nabla}\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$. We can write,

$$\boldsymbol{\nabla}\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=\boldsymbol{\nabla}\times\mathbf{E}+\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{A})=0 \quad\Rightarrow\quad \boldsymbol{\nabla}\times\left(\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right)=0 \tag{1.4.16}$$

So the curl of $\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$ always vanishes, which means we can write it in terms of the gradient of a scalar function ϕ,

$$\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}=-\boldsymbol{\nabla}\phi \qquad \text{or} \qquad \boxed{\mathbf{E}=-\boldsymbol{\nabla}\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}} \tag{1.4.17}$$

We thus can write both $\mathbf{B}$ and $\mathbf{E}$ in terms of a vector potential $\mathbf{A}$ and a scalar potential ϕ. Unlike in statics, where $\mathbf{E}$ is completely specified by ϕ, in the general case $\mathbf{E}$ is given by both ϕ and $\mathbf{A}$.

The above expression for $\mathbf{E}$ and $\mathbf{B}$ in terms of ϕ and $\mathbf{A}$ by construction solve the homogenous Maxwell's equations. We now turn to the inhomogeneous Maxwell's equation.

Substituting for $\mathbf{E}$ in terms of ϕ and $\mathbf{A}$, Gauss' Law for the electric field becomes,

$$\boldsymbol{\nabla}\cdot\mathbf{E}=-\nabla^2\phi-\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{A})=4\pi\rho \tag{1.4.18}$$

or

$$\boxed{\nabla^2\phi+\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{A})=-4\pi\rho} \tag{1.4.19}$$

Substituting for $\mathbf{E}$ and $\mathbf{B}$ in terms of ϕ and $\mathbf{A}$, Ampere's Law becomes,

$$\boldsymbol{\nabla}\times\mathbf{B}=\frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{\partial E}{\partial t} \quad\Rightarrow\quad \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})=\frac{4\pi}{c}\mathbf{j}-\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{\nabla}\phi-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} \tag{1.4.20}$$

using $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})=-\nabla^2\mathbf{A}+\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})$ we then get,

$$\boxed{-\nabla^2\mathbf{A}+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=\frac{4\pi}{c}\mathbf{j}-\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}\right)} \tag{1.4.21}$$

Gauge Invariance

As before, we can always construct $\mathbf{A}'=\mathbf{A}+\boldsymbol{\nabla}\chi$, for any scalar function χ, and $\boldsymbol{\nabla}\times\mathbf{A}'=\boldsymbol{\nabla}\times\mathbf{A}=\mathbf{B}$. But since Eq. (1.4.17) shows that now $\mathbf{A}$ enters the expression for $\mathbf{E}$, if we change $\mathbf{A}$ to $\mathbf{A}'$, we must make some corresponding change ϕ to ϕ' so that $\mathbf{E}$ does not change. The transformation that does this is called a *gauge transformation*, and is given by,

$$\boxed{\mathbf{A}'=\mathbf{A}+\boldsymbol{\nabla}\chi, \qquad \phi'=\phi-\frac{1}{c}\frac{\partial\chi}{\partial t}} \tag{1.4.22}$$

For any scalar χ, the above $\mathbf{A}'$ and ϕ' give the same $\mathbf{E}$ and $\mathbf{B}$ as do $\mathbf{A}$ and ϕ.

proof:

$$\boldsymbol{\nabla}\times\mathbf{A}'=\boldsymbol{\nabla}\times\mathbf{A}+\boldsymbol{\nabla}\times\boldsymbol{\nabla}\chi=\boldsymbol{\nabla}\times\mathbf{A}+0=\mathbf{B} \tag{1.4.23}$$

$$\left(-\boldsymbol{\nabla}\phi' - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t}\right) = -\boldsymbol{\nabla}\phi + \frac{1}{c}\boldsymbol{\nabla}\frac{\partial\chi}{\partial t} - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{\nabla}\chi = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \mathbf{E} \tag{1.4.24}$$

where the terms $\boldsymbol{\nabla}\dfrac{\partial\chi}{\partial t}$ and $\dfrac{\partial}{\partial t}\boldsymbol{\nabla}\chi$ cancel because it does not matter what order one takes the derivatives in.

As before, we can *fix the gauge* by imposing some additional constraint on $\mathbf{A}$ and ϕ that will make the inhomogeneous Maxwell's equations (1.4.19) and (1.4.21) look simpler. There are two common choices, the Lorenz gauge, and the Coulomb gauge.

Lorenz Gauge

For the Lorenz Gauge we impose the gauge constraint:

$$\boxed{\frac{1}{c}\frac{\partial\phi}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{A} = 0} \tag{1.4.25}$$

With this choice, Gauss' Law of Eq. (1.4.19) becomes

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{A}) = -4\pi\rho \qquad \Rightarrow \qquad \nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right) = -4\pi\rho \tag{1.4.26}$$

or

$$\boxed{\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho} \tag{1.4.27}$$

This is just an inhomogenous wave equation for ϕ, with ρ as the source term.

Ampere's Law of Eq. (1.4.21) becomes

$$-\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \frac{4\pi}{c}\mathbf{j} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) = \frac{4\pi}{c}\mathbf{j} - 0 \tag{1.4.28}$$

or

$$\boxed{\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j}} \tag{1.4.29}$$

This is just the inhomogenous wave equation for $\mathbf{A}$, with $\mathbf{j}$ as the source term.

Thus in the Lorenz gauge, both ϕ and $\mathbf{A}$ obey equations that have the same structure; they are both the inhomogenous wave equation. We often write the combination of operators that appears in the wave equation as $-\nabla^2 + \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} \equiv \Box^2$, called the d'Alambertian operator (note, some texts define $\Box^2$ as the negative of this, and some use the symbol $\Box$ rather than $\Box^2$). Thus we can write for the inhomogeneous Maxwell's equations, in terms of the potentials in the Lorenz gauge, as

$$\boxed{\Box^2\phi = 4\pi\rho\,, \qquad \Box^2\mathbf{A} = \frac{4\pi}{c}\mathbf{j}} \tag{1.4.30}$$

When $\rho = 0$ and $\mathbf{j} = 0$, the solutions to these equations are free propagating waves.

- Discussion Question 1.4.1

How do we know that it is always possible to find potentials ϕ and $\mathbf{A}$ so that they give the desired fields $\mathbf{E}$ and $\mathbf{B}$ *and* satisfy the Lorenz constraint $\frac{1}{c}\frac{\partial\phi}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{A} = 0$? (1.4.31)

Note, even if we have potentials that satisfy the Lorenz constraint, that does not uniquely determine the potentials. If one has $\mathbf{A}$ and ϕ obeying the Lorenz gauge constraint, and one makes a gauge transformation,

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi \qquad \phi' = \phi - \frac{1}{c}\frac{\partial\chi}{\partial t} \tag{1.4.32}$$

then $\mathbf{A}'$ and ϕ' will also be in the Lorenz gauge if $\Box^2\chi = 0$, i.e., if χ is a solution to the homogeneous wave equation.

Coulomb Gauge

In the Coulomb Gauge we make the same constraint on the potentials as we did in magnetostatics,

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{A} = 0} \tag{1.4.33}$$

In the Coulomb gauge, Gauss' Law of Eq. (1.4.19) becomes simply,

$$\boxed{\nabla^2\phi = -4\pi\rho \qquad \text{Poisson's equation!}} \tag{1.4.34}$$

ϕ satisfies the same equation as in electrostatics! Therefore the solution for ϕ is

$$\phi(\mathbf{r},t) = \int d^3r'\, \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} \tag{1.4.35}$$

No matter what is the motion of the charges, or what is the time dependence of the source $\rho(\mathbf{r},t)$, ϕ is given by the instantaneous Coulomb potential, even though electromagnetic fields have a finite velocity of propagation c.

- Discussion Question 1.4.2

In Eq. (1.4.35), the potential ϕ at position $\mathbf{r}$ is instantaneously determined by the charge density ρ at positions $\mathbf{r}'$ that are a finite distance away. Is this spooky action-at-a-distance? Does it worry you that *causality* seems to be violated? Information at $\mathbf{r}'$ seems to be instantly conveyed to the distant point $\mathbf{r}$, even though nothing can travel faster than the speed of light. (1.4.36)

In the Coulomb gauge, Ampere's Law of Eq. (1.4.21) becomes,

$$-\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \frac{4\pi}{c}\mathbf{j} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) \quad\Rightarrow\quad \Box^2\mathbf{A} = \frac{4\pi}{c}\mathbf{j} - \frac{1}{c}\boldsymbol{\nabla}\left(\frac{\partial\phi}{\partial t}\right) \quad \text{since } \boldsymbol{\nabla}\cdot\mathbf{A} = 0 \tag{1.4.37}$$

Now from Eq. (1.4.35) we have

$$\boldsymbol{\nabla}\left(\frac{\partial\phi}{\partial t}\right) = \boldsymbol{\nabla}\left[\int d^3r'\, \frac{\partial\rho(\mathbf{r}',t)}{\partial t}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right] = \boldsymbol{\nabla}\left[\int d^3r'\, \frac{\boldsymbol{\nabla}'\cdot\mathbf{j}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\right] \tag{1.4.38}$$

where in the last step we used charge conservation, $\partial\rho/\partial t = -\boldsymbol{\nabla}\cdot\mathbf{j}$.

To understand the meaning of this term, recall (and we will soon demonstrate explicity) that *any* vector function $\mathbf{f}(\mathbf{r},t)$ can always be written as the sum of a curlfree part and a divergenceless part

$$\mathbf{f} = \mathbf{f}_{\|} + \mathbf{f}_{\perp} \quad \text{where} \quad \mathrm{f}_{\|} \text{ is curlfree with } \boldsymbol{\nabla}\times\mathbf{f}_{\|} = 0, \text{ and } \mathbf{f}_{\perp} \text{ is is divergenceless with } \boldsymbol{\nabla}\cdot\mathrm{f}_{\perp} = 0 \tag{1.4.39}$$

When $\boldsymbol{\nabla}\cdot\mathbf{f}$ and $\boldsymbol{\nabla}\times\mathbf{f}$ are localized functions that vanish as $|\mathbf{r}|\to\infty$, we have for the curlfree and divergenceless parts (proof to follow),

$$\mathbf{f}_{\|}(\mathbf{r}) = -\frac{1}{4\pi}\boldsymbol{\nabla}\left[\int d^3r'\,\frac{\boldsymbol{\nabla}'\cdot\mathbf{f}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] \quad \text{and} \quad \mathbf{f}_{\perp}(\mathbf{r}) = \frac{1}{4\pi}\boldsymbol{\nabla}\times\left[\int d^3r'\,\frac{\boldsymbol{\nabla}'\times\mathbf{f}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] \tag{1.4.40}$$

The *curlfree* part is also called the *longitudinal* part of the vector function, the *divergenceless* part is also called the *transverse* part. You will see why in the first problem set.

Comparing Eq. (1.4.38) with (1.4.40), we see that $\boldsymbol{\nabla}\left(\dfrac{\partial\phi}{\partial t}\right) = 4\pi\,\mathbf{j}_{\|}$. From that we conclude from Eq. (1.4.37),

$$\Box^2\mathbf{A} = \frac{4\pi}{c}\mathbf{j} - \frac{4\pi}{c}\mathbf{j}_{\|} = \frac{4\pi}{c}\mathbf{j}_{\perp} \tag{1.4.41}$$

So in the Coulomb gauge, $\mathbf{A}$ obeys the inhomogeneous wave equation with the transverse part of the current $\mathbf{j}_{\perp}$ as the source,

$$\boxed{\Box^2\mathbf{A} = \frac{4\pi}{c}\mathbf{j}_{\perp}} \tag{1.4.42}$$

Since $\mathbf{A}$ is determined by the transverse part of the current $\mathbf{j}_{\perp}$, we can say that $\mathbf{A}$ describes the transverse modes of the fields; these are related to electromagnetic radiation (recall, for electromagnetic waves, the fields are oriented transverse to the direction of propagation). Since $\boldsymbol{\nabla}(\partial\phi/\partial t) = 4\pi\,\mathbf{j}_{\|}$, we see that ϕ is related to the longitudinal part of the current $\mathbf{j}_{\|}$; we can say that ϕ describes the longitudinal modes of the fields.

Note, the Coulomb gauge is not *Lorentz invariant* – if $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ in one inertial frame of reference, in general $\boldsymbol{\nabla}\cdot\mathbf{A} \neq 0$ in another. We will see later that the Lorenz gauge is *Lorentz invariant* – if the potential obey the Lorenz constraint in one inertial frame of reference, they obey it is all inertial frames of reference.

In the Coulomb gauge, if $\rho = 0$ then $\phi = 0$ and from Eq. (1.4.17) we have $\mathbf{E} = -\dfrac{1}{c}\dfrac{\partial\mathbf{A}}{\partial t}$.

Transverse and Longitudinal Parts of a Vector Function

To prove the preceding Eq. (1.4.40) for the transverse (divergenceless) and longitudinal (curlfree) parts of a vector function, we digress to return to Helmholtz's Theorem. This time we will prove Helmholtz's Theorem by direct construction.

Suppose one knows the divergence and curl of a vector function $\mathbf{f}(\mathbf{r})$, then we will show that this information is enough to construct $\mathbf{f}(\mathbf{r})$. We have

$$\boldsymbol{\nabla}\cdot\mathbf{f}(\mathbf{r}) = 4\pi D(\mathbf{r}) \quad \text{and} \quad \boldsymbol{\nabla}\times\mathbf{f}(\mathbf{r}) = 4\pi\mathbf{C}(\mathbf{r}) \tag{1.4.43}$$

where $D(\mathbf{r})$ and $\mathbf{C}(\mathbf{r})$ are known scalar and vector functions. We will assume the boundary condition that $\mathbf{f}(\mathbf{r})\to 0$ as $|\mathbf{r}|\to\infty$.

Let us assume a solution for $\mathbf{f}$ of the form

$$\mathbf{f} = -\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\times\mathbf{W} \tag{1.4.44}$$

where ϕ is a scalar function and $\mathbf{W}$ is a vector function. Now consider the divergence of $\mathbf{f}$.

$$\boldsymbol{\nabla}\cdot\mathbf{f} = -\nabla^2\phi + \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{W}) = -\nabla^2\phi + 0 = 4\pi D(\mathbf{r}) \qquad \text{since } \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{W}) = 0 \text{ for any } \mathbf{W}. \tag{1.4.45}$$

So

$$\nabla^2\phi = -4\pi D(\mathbf{r}) \qquad \text{this is just Poisson's equation like we saw in electrostatics} \tag{1.4.46}$$

For the boundary condition $\phi(\mathbf{r})\to 0$ as $|\mathbf{r}|\to\infty$, the solution is then,

$$\phi(\mathbf{r}) = \int d^3r'\,\frac{D(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{1.4.47}$$

Now consider the curl of $\mathbf{f}$.

$$\boldsymbol{\nabla}\times\mathbf{f} = -\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{W}) = 0 - \nabla^2\mathbf{W} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{W}) = 4\pi\mathbf{C}(\mathbf{r}) \tag{1.4.48}$$

Let us choose a gauge for $\mathbf{W}$ so that $\boldsymbol{\nabla}\cdot\mathbf{W} = 0$. We can do this just like we can do it in magnetostatics. Then the above becomes

$$\nabla^2\mathbf{W} = -4\pi\mathbf{C}(\mathbf{r}) \tag{1.4.49}$$

and the solution, assuming $\mathbf{W}\to 0$ as $|\mathbf{r}|\to\infty$, is

$$\mathbf{W}(\mathbf{r}) = \int d^3r'\,\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{1.4.50}$$

So we have constructed a solution that has the desired divergence and curl,

$$\mathbf{f}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}) + \boldsymbol{\nabla}\times\mathbf{W}(\mathbf{r}) = -\boldsymbol{\nabla}\left[\int d^3r'\,\frac{D(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] + \boldsymbol{\nabla}\times\left[\int d^3r'\,\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] \tag{1.4.51}$$

where $\boldsymbol{\nabla}\cdot\mathbf{f} = 4\pi D$ and $\boldsymbol{\nabla}\times\mathbf{f} = 4\pi\mathbf{C}$.

Note, for the above solution to be well defined, the integrals must converge. They will converge if the "sources" $D(\mathbf{r})$ and $\mathbf{C}(\mathbf{r})$ are sufficiently localized that $D(\mathbf{r})\to 0$ and $\mathbf{C}(\mathbf{r})\to 0$ sufficiently fast as $|\mathbf{r}|\to\infty$.

Now we will show that the above solution we constructed is the unique solution. Suppose there was another solution $\mathbf{g}(\mathbf{r})$ so that $\boldsymbol{\nabla}\cdot\mathbf{g}(\mathbf{r}) = 4\pi D(\mathbf{r})$ and $\boldsymbol{\nabla}\times\mathbf{g}(\mathbf{r}) = 4\pi\mathbf{C}(\mathbf{r})$. Then consider $\mathbf{h} = \mathbf{f} - \mathbf{g}$. We have

$$\boldsymbol{\nabla}\cdot\mathbf{h} = 0 \quad \text{and} \quad \boldsymbol{\nabla}\times\mathbf{h} = 0 \tag{1.4.52}$$

One can show that the only such $\mathbf{h}$ that also has $\mathbf{h}(\mathbf{r})\to 0$ as $|\mathbf{r}|\to\infty$ is $\mathbf{h} = 0$, so we conclude that $\mathbf{g} = \mathbf{f}$, and our solutions is unique.

As a consequence of Helmholtz's Theorem we have also shown the following:

(1) Any vector function $\mathbf{f}(\mathbf{r})$ can be written in terms of a scalar and vector potential

$$\mathbf{f} = -\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\times\mathbf{W} \tag{1.4.53}$$

or equivalently

(2) Any vector function $\mathbf{f}$ can be written in terms of a curlfree and a divergenceless part

$$\mathbf{f} = \mathbf{f}_{\|} + \mathbf{f}_{\perp} \qquad \text{where } \boldsymbol{\nabla}\times\mathbf{f}_{\|} = 0 \text{ and } \boldsymbol{\nabla}\cdot\mathbf{f}_{\perp} = 0 \tag{1.4.54}$$

where

$$\mathbf{f}_{\|}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}) = -\boldsymbol{\nabla}\left[\frac{1}{4\pi}\int d^3r'\,\frac{[\boldsymbol{\nabla}'\cdot\mathbf{f}(\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}|}\right] \tag{1.4.55}$$

and

$$\mathbf{f}_{\perp}(\mathbf{r}) = \boldsymbol{\nabla} \times \mathbf{W}(\mathbf{r}) = \boldsymbol{\nabla} \times \left[\frac{1}{4\pi} \int d^3r' \, \frac{[\boldsymbol{\nabla}' \times \mathbf{f}(\mathbf{r}')]}{|\mathbf{r} - \mathbf{r}|} \right] \tag{1.4.56}$$

where in the above we used $D(\mathbf{r}') = [\boldsymbol{\nabla}' \cdot \mathbf{f}(\mathbf{r}')]/4\pi$ and $\mathbf{C}(\mathbf{r}') = [\boldsymbol{\nabla}' \times \mathbf{f}(\mathbf{r}')]/4\pi$.

$\mathbf{f}_{\|}$ is called the longitudinal part of $\mathbf{f}$ and $\mathbf{f}_{\perp}$ is called the transverse part of $\mathbf{f}$. To understand the reason for these names we need to consider the Fourier transforms of these functions. You will do that for homework!

The above can be generalized to situations where $\mathbf{f}$ satisfies other boundary conditions, such as $\mathbf{f}$ has a specified value on a given boundary surface. One just replaces the function $1/|\mathbf{r} - \mathbf{r}'|$ in the integrals by the appropriate *Green's function* corresponding to the given boundary. We will discuss more on this later.