Unit 2: Electrostatics and Magnetostatics

In this unit we consider static situations, discussing several mathematical methods to solve problems in electrostatics and magnetostatics. In the process we will see some interesting physical phenomena!

Unit 2-1: Electrostatics as a Boundary Value Problem

All of electrostatics can be viewed as solving Poisson's equation,

$$-\nabla^2\phi = 4\pi\rho \qquad \text{with} \qquad \mathbf{E} = -\boldsymbol{\nabla}\phi \qquad \text{(statics only)} \tag{2.1.1}$$

Physical Meaning of the Potential ϕ

The work done to move a test charge δq from position $\mathbf{r}_1$ to $\mathbf{r}_2$ in the presence of an electric field $\mathbf{E}$ is,

$$W_{12} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \boldsymbol{d\ell}\cdot\mathbf{F} \tag{2.1.2}$$

where $\mathbf{F}$ is the force required to move the charge. How is $\mathbf{F}$ related to $\mathbf{E}$? Since $\mathbf{E}$ exerts a force $\delta q\mathbf{E}$ on the charge, the force $\mathbf{F}$ we apply to move the charge must *counterbalance* that electric force so that we can move the charge quasistatically $\quad\Rightarrow\quad \mathbf{F} = -\delta q\mathbf{E}$. So the work we do on the charge to move it is

$$W_{12} = -\delta q\int_{\mathbf{r}_1}^{\mathbf{r}_2} \boldsymbol{d\ell}\cdot\mathbf{E} = \delta q\int_{\mathbf{r}_1}^{\mathbf{r}_1} \boldsymbol{d\ell}\cdot\boldsymbol{\nabla}\phi = \delta q\left[\phi(\mathbf{r}_2)-\phi(\mathbf{r}_1)\right] \tag{2.1.3}$$

or

$$\phi(\mathbf{r}_2)-\phi(\mathbf{r}_1) = \frac{W_{12}}{\delta q} \tag{2.1.4}$$

The difference in potential between two points is the work per unit charge needed to move a test charge between those two points. This is *only* true for static situations because $\mathbf{E} = -\boldsymbol{\nabla}\phi$ *only* in statics. More generally, $\mathbf{E}$ involves also the vector potential $\mathbf{A}$.

Green's Function

Poisson's equation is $-\nabla^2\phi = 4\pi\rho$. We already know that for a point charge q at position $\mathbf{r}'$, i.e. $\rho(\mathbf{r}) = q\delta(\mathbf{r}-\mathbf{r}')$, the solution to the above is

$$\phi(\mathbf{r}) = \frac{q}{|\mathbf{r}-\mathbf{r}'|} \qquad \Rightarrow \qquad \nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = 4\pi\delta(|\mathbf{r}-\mathbf{r}'|) \tag{2.1.5}$$

We call the special solution for a point source the *Green's function* G for the differential operator. For the Poisson equation

$$-\nabla^2 G(\mathbf{r},\mathbf{r}') = 4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{2.1.6}$$

$G(\mathbf{r},\mathbf{r})$ gives the potential at position $\mathbf{r}$ due to a unit source at position $\mathbf{r}'$. Generally one has to also specify a desired boundary condition on the Green's function on the boundary of the system.

For the Coulomb solution (2.1.5) for a point charge, the implicit boundary condition is that the boundary of the system is taken to infinity, and the potential vanishes infinitely far from the charge.

$$G(\mathbf{r},\mathbf{r}') \to 0 \quad \text{as} \quad |\mathbf{r}-\mathbf{r}'| \to \infty \tag{2.1.7}$$

If one knows the Green's function, then one can find the solution for *any* distribution of sources $\rho(\mathbf{r})$

$$\boxed{\phi(\mathbf{r}) = \int d^3r'\, G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')} \tag{2.1.8}$$

$$\text{proof:} \quad -\nabla^2\phi(\mathbf{r}) = \int d^3r' \left[-\nabla^2 G(\mathbf{r},\mathbf{r}')\right]\rho(\mathbf{r}') = \int d^3r' \left[4\pi\delta(\mathbf{r}-\mathbf{r}')\right]\rho(\mathbf{r}') = 4\pi\rho(\mathbf{r}) \tag{2.1.9}$$

where we note that the operator ∇^2 acts only on the variable $\mathbf{r}$ and *not* on $\mathbf{r}'$.

For the point charge in an infinite system, the translational symmetry of infinite empty space tells us that $G(\mathbf{r},\mathbf{r}')$ can only depend on the $|\mathbf{r}-\mathbf{r}'|$, i.e. the distance between the observer at $\mathbf{r}$ and the charge at $\mathbf{r}'$.

For a system consisting of a volume enclosed by a finite surface (the "walls"), this is no longer true. The walls of the system break the translational symmetry of space, and the potential at position $\mathbf{r}$ will depend not just on the distance between $\mathbf{r}$ and the charge, but also on the distance between $\mathbf{r}$ and the walls. In this case the Green's function will no longer be the simple Coulomb solution $1/|\mathbf{r}-\mathbf{r}'|$, but will be determined by the geometry of the bounding walls, and the boundary condition that is imposed on the walls.

If one specifies the value of the potential ϕ on the bounding walls, this is known as the *Dirichlet* boundary condition. If one specifies the value of $\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\phi = \partial\phi/\partial n$, i.e. the derivative of the potential in the direction normal to the walls (for electrostatics, this is just minus the normal component of the electric field at the walls), this is known as the *Neumann* boundary condition. The Green's function for Dirichlet/Neumann boundary conditions is in general difficult to find for a general geometry of bounding walls.

Poisson's Equation as a Boundary Value Problem

Consider a conducting sphere of radius R with net charge q on it (as $R \to 0$ this would be a point charge). What is $\phi(\mathbf{r})$? What is $\mathbf{E}(\mathbf{r})$? For this very simple problem you could probably guess the solution, or easily solve using Gauss' Law in integral form. However here we will solve this problem as a differential equation problem so as to use it as an example with which to introduce the notion of solving an electrostatic problem as a boundary value problem.

Review: Properties of conductors in electrostatics:

1) $\mathbf{E} = 0$ inside the conductor. Because if we had $\mathbf{E} \neq 0$ then a current $\mathbf{j} = \sigma\mathbf{E}$ would flow and it would not be a static situation (σ is the conductivity).

2) $\rho = 0$ inside the conductor. Because if $\mathbf{E} = 0$ then $\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho = 0$.

3) Any net charge on the conductor must lie on the surface. Because $\rho = 0$ inside the conductor.

4) $\phi =$ constant throughout the conductor. Because if $\phi \neq$ constant then $-\boldsymbol{\nabla}\phi = \mathbf{E} \neq 0$, but $\mathbf{E} = 0$ inside the conductor.

5) Just outside the conductor $\mathbf{E} \perp \hat{\mathbf{n}}$, the electric field is perpendicular to the surface. Because if $\mathbf{E}$ had a component parallel to the surface, then it would exert a force on the charges at the surface and lead to a surface current, so it would not be a static situation.

Now apply to our conducting sphere. For the conducting sphere, $\rho = 0$ for $r > R$ (outside) and $r < R$ (inside). All charge is on the surface of the sphere.

$$\Rightarrow \quad \nabla^2\phi = 0 \qquad \text{for} \qquad \begin{cases} r < R \\ r > R \end{cases} \tag{2.1.10}$$

We will thus solve Laplace's equation $\nabla^2\phi = 0$ separately inside the sphere and outside the sphere, and then match the two solutions up at the surface of the sphere $r = R$.

By the spherical symmetry of the problem – the system is invariant with respect to rotations about the center of the sphere – we expect our solution to be spherically symmetric, i.e. $\phi(\mathbf{r})$ should depend only on the radial distance from the center of the sphere $r = |\mathbf{r}|$.

We can therefore solve Laplace's equation by writing ∇^2 in spherical coordinates, keeping only the piece that involves radial derivatives.

$$\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 0 \tag{2.1.11}$$

Integrate once to get

$$r^2\frac{d\phi}{dr} = -C_0 \quad \text{with } C_0 \text{ a constant} \qquad \Rightarrow \qquad \frac{d\phi}{dr} = -\frac{C_0}{r^2} \tag{2.1.12}$$

Integrate again to get

$$\phi(r) = \frac{C_0}{r} + C_1 \quad \text{with } C_1 \text{ another constant} \tag{2.1.13}$$

So we have

$$\text{outside the sphere:} \quad \phi^{\text{out}}(r) = \frac{C_0^{\text{out}}}{r} + C_1^{\text{out}}, \quad r > R \tag{2.1.14}$$

$$\text{inside the sphere:} \quad \phi^{\text{in}}(r) = \frac{C_0^{\text{in}}}{r} + C_1^{\text{in}}, \quad r < R \tag{2.1.15}$$

where in general $C_0^{\text{out}} \neq C_0^{\text{in}}$ and $C_1^{\text{out}} \neq C_1^{\text{in}}$, since the solution outside does not necessarily go smoothly into the solution inside, because of the charge layer on the surface at $r = R$ that separates the two regions. We need to determine the unknown constants $C_0^{\text{out}}, C_0^{\text{in}}, C_1^{\text{out}}, C_1^{\text{in}}$ by applying *boundary conditions* corresponding to the physical situation.

boundary conditions

1) For $r > R$, we want $\phi \to 0$ as $r \to \infty$ – this is the boundary condition at infinity and leads to the conclusion $C_1^{\text{out}} = 0$.

$$\Rightarrow \quad \phi^{\text{out}}(r) = \frac{C_0^{\text{out}}}{r} \qquad \text{So outside } \phi^{\text{out}} \text{ has just the usual Coulomb form for a point charge!} \tag{2.1.16}$$

2) For $r < R$ (i) if we have a solid conducting sphere, then the region $r < R$ is a conductor with $\phi =$ constant so we conclude that $C_0^{\text{in}} = 0$, or (ii) if we have a charged conducting shell rather than a solid sphere, we could argue that since there is no charge at the origin, there is no reason why ϕ should diverge at the origin, so $C_0^{\text{in}} = 0$.

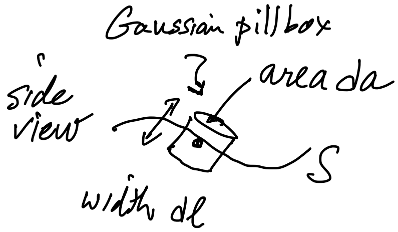
$$\Rightarrow \quad \phi^{\text{in}}(r) = C_1^{\text{in}} \qquad \text{So inside } \phi^{\text{in}} \text{ is constant} \tag{2.1.17}$$

3) Now we need the boundary condition at $r = R$ on the surface, where the "inside" and the "outside" meet.

Review: Electric field and potential at a surface charge layer

For a general surface S with surface charge density (charge per unit area) $\sigma(\mathbf{r})$ at position $\mathbf{r}$ on S, $\sigma(\mathbf{r})da$ is the total charge in area da at position $\mathbf{r}$ on the surface.

i) Take a "Gaussian pillbox" surface about point $\mathbf{r}$ on the surface S, of width $d\ell$ and area da.



Gauss' Law in integral form gives, $\oint_S da\,\hat{\mathbf{n}}\cdot\mathbf{E} = 4\pi Q_{\rm encl}$

We expect $\mathbf{E}$ is finite, so the contribution from the sides of the pillbox vanish as $d\ell \to 0$. All that remains are the integrals over the top and bottom sides of small area da. We have,

$$\Rightarrow \quad \oint_S da\,\hat{\mathbf{n}}\cdot\mathbf{E} = \int_{\rm top} da\,\hat{\mathbf{n}}\cdot\mathbf{E} + \int_{\rm bottom} da\,\hat{\mathbf{n}}\cdot\mathbf{E} = \left(\hat{\mathbf{n}}^{\rm top}\cdot\mathbf{E}^{\rm top} + \hat{\mathbf{n}}^{\rm bottom}\cdot\mathbf{E}^{\rm bottom}\right) da \tag{2.1.18}$$

where we have approximated the integral by the integrand evaluated at $\mathbf{r}$ times the area da. This is good since da is sufficiently small that we can assume $\mathbf{E}$ is roughly constant over it. $\mathbf{E}^{\rm top}$ is the electric field at point $\mathbf{r}$ just above the surface S, while $\mathbf{E}^{\rm bottom}$ is the electric field at point $\mathbf{r}$ just below the surface S.

Now if $\hat{\mathbf{n}}^{\rm top} \equiv \hat{\mathbf{n}}$ is the outward pointing normal to S on the top of the surface, then $\hat{\mathbf{n}}^{\rm bottom} = -\hat{\mathbf{n}}$ is the outward pointing normal on the bottom of the surface. We then get

$$\oint_S da\,\hat{\mathbf{n}}\cdot\mathbf{E} = \left(\mathbf{E}^{\rm top} - \mathbf{E}^{\rm bottom}\right)\cdot\hat{\mathbf{n}}\,da = 4\pi Q_{\rm encl} = 4\pi\sigma(\mathbf{r})da \tag{2.1.19}$$

so

$$\boxed{\left(\mathbf{E}^{\rm top} - \mathbf{E}^{\rm bottom}\right)\cdot\hat{\mathbf{n}} = 4\pi\sigma(\mathbf{r})} \tag{2.1.20}$$

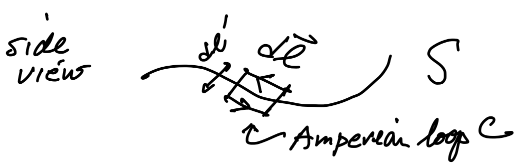
The normal component of $\mathbf{E}$ is *discontinuous* at the charged surface, with a jump equal to $4\pi\sigma$.

We can rewrite the above in terms of the potential. With $\mathbf{E} = -\boldsymbol{\nabla}\phi$ the above becomes,

$$\left(-\boldsymbol{\nabla}\phi^{\rm top} + \boldsymbol{\nabla}\phi^{\rm bottom}\right)\cdot\hat{\mathbf{n}} = \boxed{-\frac{\partial\phi^{\rm top}}{\partial n} + \frac{\partial\phi^{\rm bottom}}{\partial n} = 4\pi\sigma} \tag{2.1.21}$$

where $\partial\phi/\partial n = \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\phi$ is the directional derivative in the normal direction.

ii) Now take an "Amperian loop" C about point $\mathbf{r}$ on the surface S, of length $d\ell$ and width $d\ell'$.



$$\boldsymbol{\nabla}\times\mathbf{E} = 0 \quad \Rightarrow \quad \oint_C \boldsymbol{d\ell}\cdot\mathbf{E} = 0 \tag{2.1.22}$$

Since $\mathbf{E}$ should be finite at the surface, the contribution to the integral from the sides of the loop perpendicular to the surface will vanish if we take $d\ell' \to 0$. This leaves only the contribution from the sides parallel to the surface, which gives

$$\Rightarrow \quad \oint_C \boldsymbol{d\ell}\cdot\mathbf{E} = \boxed{\left(\mathbf{E}^{\rm top} - \mathbf{E}^{\rm bottom}\right)\cdot\boldsymbol{d\ell} = 0} \tag{2.1.23}$$

Here $\boldsymbol{d\ell}$ is any infinitesmal tangent vector to the surface at $\mathbf{r}$. Since $\boldsymbol{d\ell}$ can point in any direction tangent to the surface, we conclude that the tangential component of $\mathbf{E}$ must be *continuous* at the surface. Combining Eqs. (2.1.20) and (2.1.23) gives,

$$\boxed{\mathbf{E}^{\rm top} - \mathbf{E}^{\rm bottom} = 4\pi\sigma(\mathbf{r})\,\hat{\mathbf{n}}} \tag{2.1.24}$$

iii) $\mathbf{E} = -\boldsymbol{\nabla}\phi \quad \Rightarrow \quad \phi(\mathbf{r}_2) - \phi(\mathbf{r}_1) = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \boldsymbol{d\ell}\cdot\mathbf{E}.$

If we take $\mathbf{r}_2$ just above the point $\mathbf{r}$ on the surface, and $\mathbf{r}_1$ just below the point $\mathbf{r}$ on the surface, and take the distance between these two points $d\ell \to 0$, then since we expect $\mathbf{E}$ is finite at the surface, we conclude

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} d\boldsymbol{\ell}\cdot\mathbf{E} \to 0 \quad \Rightarrow \quad \boxed{\phi^{\rm top} = \phi^{\rm bottom}} \tag{2.1.25}$$

The potential ϕ is *continuous* at the charged surface.

We can now apply the results of Eqs. (2.1.25) and (2.1.21) to the conducting sphere.

$$\phi \text{ is continuous} \quad \Rightarrow \quad \phi^{\rm in}(R) = \phi^{\rm out}(R) \quad \Rightarrow \quad C_1^{\rm in} = \frac{C_0^{\rm out}}{R} \tag{2.1.26}$$

Now only one unknown is left. Use the condition that the potential has a discontinuous normal derivative,

$$-\frac{\partial\phi^{\rm top}}{\partial n} + \frac{\partial\phi^{\rm bottom}}{\partial n} = 4\pi\sigma \tag{2.1.27}$$

With $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ the radial direciton, and "top" = "out" and "bottom" = "in", we have,

$$\left[-\frac{d\phi^{\rm out}}{dr} + \frac{d\phi^{\rm in}}{dr}\right]_{r=R} = 4\pi\sigma \tag{2.1.28}$$

But $\dfrac{d\phi^{\rm in}}{dr} = 0$ as $\phi^{\rm in}$ is constant. So

$$-\left.\frac{\phi^{\rm out}}{dr}\right|_{r=R} = 4\pi\sigma \quad \Rightarrow \quad -\frac{d}{dr}\left.\left(\frac{C_0^{\rm out}}{r}\right)\right|_{r=R} = \frac{C_0^{\rm out}}{R^2} = 4\pi\sigma = 4\pi\left(\frac{q}{4\pi R^2}\right) = \frac{q}{R^2} \tag{2.1.29}$$

where by rotational symmetry we used that σ is uniform over the surface and so $\sigma = q/4\pi R^2$.

So finally we have from the two boundary conditions at the surface that $C_0^{\rm out} = q$ and $C_1^{\rm in} = C_0^{\rm out}/R = q/R$. Our solution for the potential is thus,

$$\phi(r) = \begin{cases} \dfrac{q}{R} & r < R \text{ inside} \\[2ex] \dfrac{q}{r} & r > R \text{ outside} \end{cases} \tag{2.1.30}$$

For the electric field we get

$$\mathbf{E} = -\boldsymbol{\nabla}\phi = -\frac{d\phi}{dr}\hat{\mathbf{r}} = \begin{cases} 0 & r < R \text{ inside} \\[2ex] \dfrac{q}{r^2}\hat{\mathbf{r}} & r > R \text{ outside} \end{cases} \tag{2.1.31}$$

Outside the sphere, this is just the familiar Coulomb solution for a point charge at the origin!

Summary: We can view the preceding solution for $\phi^{\rm out}$ as solving Laplace's equation $\nabla^2\phi = 0$ subject to a specified boundary condition on the normal derivative of ϕ at the boundary $r = R$ of the "outside" region of the system.

An Alternative Problem: Another physical situation would be to connect the conducting sphere to a battery that charges the sphere to a fixed voltage ϕ_0 (in statvolts!) with respect to the ground $\phi = 0$ at $r \to \infty$. What now is the solution for ϕ?

As before, outside the sphere we must have $\phi = \dfrac{C_0}{r}$. But now the boundary condition is to specify the value of ϕ on the boundary of the "outside" region, i.e. $\phi(R) = \phi_0$. We therefore have,

$$\phi(R) = \phi_0 \quad \Rightarrow \quad \frac{C_0}{R} = \phi_0 \quad \Rightarrow \quad C_0 = \phi_0 R \tag{2.1.32}$$

so

$$\phi(r) = \phi_0 \frac{R}{r} \tag{2.1.33}$$

Comparing this to the original version of the problem, we see that charging the sphere to the voltage ϕ_0 induces a net charge $q = \phi_0 R$ on the sphere.

These two versions of the conducting sphere problem are examples of a more general boundary value problem: solve $\nabla^2\phi = 0$ in a give region of space, subject to one of the following two types of boundary conditions on the bounding surfaces of the region,

i) Neumann boundary condition: $\dfrac{d\phi}{dn}$ – the normal derivative of ϕ is specified on the bounding surface.

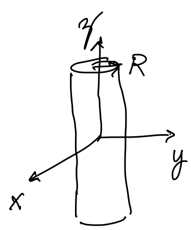
ii) Dirichlet boundary condition: ϕ – the value of ϕ is specified on the bounding surface.

If the bounding surface consists of disjoint pieces, it is possible to specify either (i) or (ii) on each piece separately to get a mixed boundary value problem.

Additional Examples

It is generally only possible to get simple analytical solutions when the boundary surfaces have a simple geometry. Our previous problem involved a geometry with spherical symmetry. Here we will consider two other simple geometries, one with cylindrical symmetry, and one with planar symmetry.

1) **Cylindrical Symmetry**: Consider an infinite conducting cylindrical wire of radius R, oriented along the $\hat{\mathbf{z}}$ axis. The wire has a uniform line charge λ (charge per unit length).



The surface charge on the surface of the wire is then $\sigma = \dfrac{\lambda}{2\pi R}$.

We expect the potential ϕ to have cylindrical symmetry, i.e. rotational symmetry about the $\hat{\mathbf{z}}$ and translational symmetry along the $\hat{\mathbf{z}}$ axis. If we express our problem in terms of cylindrical coordinates (r, φ, z), then ϕ should depend only on the cylindrical radial coordinate r.

$\nabla^2\phi = 0$ for $r > R$ and $r < R$. To solve we will write ∇^2 in cylindrical coordinates. Since ϕ depends only on r, we keep only the piece that involves the cylindrical radial derivatives.

$$\nabla^2\phi = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0 \tag{2.1.34}$$

Integrate to get

$$r\frac{d\phi}{dr} = C_0 \qquad \text{with } C_0 \text{ a constant} \quad \Rightarrow \quad \frac{d\phi}{dr} = \frac{C_0}{r} \tag{2.1.35}$$

Integrate again to get

$$\phi(r) = C_0 \ln r + C_1 \qquad \text{with } C_1 \text{ a constant} \tag{2.1.36}$$

Note, one cannot now choose $\phi \to 0$ as $r \to \infty$. Physically, this happens because in this problem the charge is not localized but extends off to infinity since the wire is infinitely long. One needs to fix the zero of the potential at some other radius. A convenient choice is to choose $\phi = 0$ at $r = R$, but any other choice could also be made since adding a constant to ϕ does not change the value of $\mathbf{E} = -\boldsymbol{\nabla}\phi$.

We have

$$\phi^{\text{out}} = C_0^{\text{out}} \ln r + C_1^{\text{out}} \quad \text{for } r > R \qquad \text{and} \quad \phi^{\text{in}} = C_0^{\text{in}} \ln r + C_1^{\text{in}} \quad \text{for } r < R. \tag{2.1.37}$$

If the wire is a solid conductor, then we have $\phi^{\rm in}=$ constant inside the conducting wire, so $C_0^{\rm in}=0$. Or if the wire is a thin cylindrical conducting shell we could argue that $\phi^{\rm in}$ should not diverge as $r\to 0$, so again we must have $C_0^{\rm in}=0$.

So inside, $\phi^{\rm in}=C_1^{\rm in}$ is a constant.

Now we apply the boundary conditions at $r=R$. The discontinuity in the normal derivative of ϕ at the boundary gives,

$$\left[-\frac{d\phi^{\rm out}}{dr}+\frac{d\phi^{\rm in}}{dr}\right]_{r=R}=4\pi\sigma \quad\Rightarrow\quad -\frac{C_0^{\rm out}}{R}=4\pi\sigma=4\pi\left(\frac{\lambda}{2\pi R}\right)=\frac{2\lambda}{R} \quad\Rightarrow\quad C_0^{\rm out}=-2\lambda \tag{2.1.38}$$

The continuity of ϕ at the boundary gives

$$\phi^{\rm in}(R)=\phi^{\rm out}(R) \quad\Rightarrow\quad C_1^{\rm in}=-2\lambda\ln R+C_1^{\rm out} \tag{2.1.39}$$

The remaining unknown is $C_1^{\rm out}$. Its value is not too important as it is just an additive constant to both $\phi^{\rm in}$ and $\phi^{\rm out}$, and so does not effect the electric field $\mathbf{E}=-\boldsymbol{\nabla}\phi$.

If we set the zero of the potential to be at $r=R$, then we have,

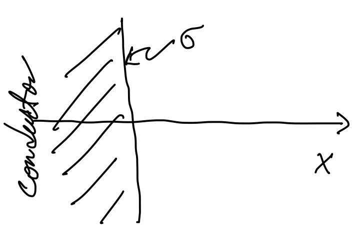
$$\phi^{\rm out}(R)=-2\lambda\ln R+C_1^{\rm out}=0 \quad\Rightarrow\quad C_1^{\rm out}=2\lambda\ln R \tag{2.1.40}$$

and we then have

$$\phi(r)=\left\{\begin{array}{cc} -2\lambda\ln(r/R) & r>R \\ 0 & r<R\end{array}\right. \quad\Rightarrow\quad -\boldsymbol{\nabla}\phi=-\frac{d\phi}{dr}\hat{\mathbf{r}}=\mathbf{E}=\left\{\begin{array}{cc} \dfrac{2\lambda}{r}\hat{\mathbf{r}} & r>R \\ & \\ 0 & r<R\end{array}\right. \tag{2.1.41}$$

Unlike for a point charge where $E\sim 1/r^2$, here the electric field decays as $E\sim 1/r$, the inverse of the *cylindrical* radial distance.

2) **Planar Symmetry:** Consider an infinite conducting half space, with uniform surface charge σ on the flat surface which is the yz plane at $x=0$.



Since the system has translational symmetry in the y and z directions, we expect that ϕ depends only on the coordinate x. In Cartesian coordinates we can then write Laplace's equation as,

$$\nabla^2\phi=\frac{d^2\phi}{dx^2}=0 \quad\Rightarrow\quad \left\{\begin{array}{cc} \phi^+(x)=c_0^+x+c_1^+ & x>0 \\ & \\ \phi^-(x)=c_0^-x+c_1^- & x<0\end{array}\right. \tag{2.1.42}$$

For $x<0$, the space is filled with the conductor so $\phi^-=$ constant $\Rightarrow$ $c_0^-=0$.

At $x=0$, ϕ is continuous $\Rightarrow$ $\phi^-(0)=\phi^+(0)$ $\Rightarrow$ $c_1^-=c_1^+$.

At $x=0$, $\dfrac{d\phi}{dx}$ is discontinuous $\Rightarrow$ $\left[-\dfrac{d\phi^+}{dx}+\dfrac{d\phi^-}{dx}\right]_{x=0}=4\pi\sigma$ $\Rightarrow$ $-\left.\dfrac{d\phi^+}{dx}\right|_{x=0}=4\pi\sigma$ $\Rightarrow$ $c_0^+=-4\pi\sigma$.

So we conclude

$$\phi(x)=\left\{\begin{array}{cc} -4\pi\sigma x+c_1^+ & x>0 \\ & \\ c_1^+ & x<0\end{array}\right. \tag{2.1.43}$$

Note, c_1^+ is a constant, and so its value does not effect the electric field $\mathbf{E}=-\boldsymbol{\nabla}\phi$. We cannot choose the constant to make $\phi\to 0$ as $x\to 0$. If we choose ϕ to vanish at $x=0$, then $c_1^+=0$. The electric field is

$$-\boldsymbol{\nabla}\phi=\mathbf{E}=\left\{\begin{array}{cc} -4\pi\sigma\,\hat{\mathbf{x}} & x>0 \\ & \\ 0 & x<0\end{array}\right. \tag{2.1.44}$$

The field is zero inside the conductor, and uniform to the right of the conductor

A related problem is that of an infinite flat conducting plane. Again the yz plane at $x = 0$ has a uniform surface charge σ, but now the region $x < 0$ is empty space just like the region $x > 0$.

Again the solution for ϕ must have the same form as in Eq. (2.1.42). Again, continuity of ϕ at $x = 0$ requires $c_1^+ = c_1^-$.

But now, since ϕ^- is not necessarily a constant, the discontinuity condition on $\dfrac{d\phi}{dx}$ becomes,

$$\left[-\frac{d\phi^+}{dx} + \frac{d\phi^-}{dx}\right]_{x=0} = 4\pi\sigma \quad \Rightarrow \quad -c_0^+ + c_0^- = 4\pi\sigma \tag{2.1.45}$$

Let us define the average of c_0^+ and c_0^- as $\bar{c}_0 = \dfrac{c_0^+ + c_0^-}{2}$. With $\dfrac{c_0^+ - c_0^-}{2} = -2\pi\sigma$ we can write

$$c_0^- = \bar{c}_0 + 2\pi\sigma \qquad \text{and} \qquad c_0^+ = \bar{c}_0 - 2\pi\sigma \tag{2.1.46}$$

so

$$\phi = \begin{cases} -2\pi\sigma\, x + \bar{c}_0 x + c_1^+ & x > 0 \\ \\ 2\pi\sigma\, x + \bar{c}_0 x + c_1^+ & x < 0 \end{cases} \tag{2.1.47}$$

and the electric field is

$$-\boldsymbol{\nabla}\phi = \mathbf{E} = \begin{cases} (2\pi\sigma - \bar{c}_0)\,\hat{\mathbf{x}} & x > 0 \\ \\ (-2\pi\sigma - \bar{c}_0)\,\hat{\mathbf{x}} & x < 0 \end{cases} \tag{2.1.48}$$

The constant c_1^+ does not effect the value of $\mathbf{E}$ since it is just an additive constant to ϕ. If we wish to choose ϕ such that $\phi(0) = 0$, then we get $c_1^+ = 0$.

The constant $\bar{c}_0$ represents a uniform constant electric field $\mathbf{E}_0 = -\bar{c}_0\hat{\mathbf{x}}$ filling all of space, that exists independently of the charged surface (it remains even if we take $\sigma \to 0$). If we assume that the electric field we are interested in is the field arising from the σ on the plane, then we can set $\bar{c}_0 = 0$. Equivalently, if the plane is the only source of $\mathbf{E}$, then we expect by reflection symmetry about the yz plane that ϕ should depend only on $|x|$, which would then require $c_0^- = -c_0^+$ and again we get $\bar{c}_0 = 0$. In this case (taking also $c_1^+ = 0$), we have,

$$\phi = \begin{cases} -2\pi\sigma\, x & x > 0 \\ \\ 2\pi\sigma\, x & x < 0 \end{cases} \qquad \Rightarrow \qquad -\boldsymbol{\nabla}\phi = \mathbf{E} = \begin{cases} 2\pi\sigma\,\hat{\mathbf{x}} & x > 0 \\ \\ -2\pi\sigma\,\hat{\mathbf{x}} & x < 0 \end{cases} \tag{2.1.49}$$

$\mathbf{E}$ has constant magnitude but is oppositely directed on either side of the charged plane.