Unit 2-3: Separation of Variables

Our next special method for solving Poisson's equation is the method of separation of variables. As with the image charge method, separation of variables only works well for certain simple geometries.

Rectangular Coordinates

If the system has a rectangular boundary, and contains no charge, we can look for solutions to $\nabla^2\phi = 0$ of the form,

$$\phi(\mathbf{r}) = X(x)Y(y)Z(z) \qquad \text{product of three functions, each of which depends on one variable only} \tag{2.3.1}$$

Then,

$$\nabla^2\phi = 0 \quad \Rightarrow \quad \frac{1}{\phi}\nabla^2\phi = 0 \quad \Rightarrow \quad \frac{1}{X(x)}\frac{d^2X}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z}{dz^2} = 0 \tag{2.3.2}$$

The only way this can be equal to zero for *all* values of x, y, and z, is if each of the three terms is a constant. Call them a^2, b^2, and c^2. Then we have,

$$\frac{1}{X}\frac{d^2X}{dx^2} = a^2 \qquad \Rightarrow \qquad X(x) = A_1\mathrm{e}^{-ax} + A_2\mathrm{e}^{ax} \tag{2.3.3}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = b^2 \qquad \Rightarrow \qquad Y(y) = B_1\mathrm{e}^{-by} + B_2\mathrm{e}^{by} \tag{2.3.4}$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = c^2 \qquad \Rightarrow \qquad Z(z) = C_1\mathrm{e}^{-cz} + C_2\mathrm{e}^{cz} \tag{2.3.5}$$

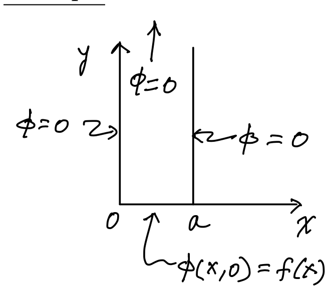
with $a^2 + b^2 + c^2 = 0 \;\Rightarrow\;$ at least one of the a^2, b^2, c^2 must be negative $\;\Rightarrow\;$ at least one of the a, b, c is an imaginary number.

Above is one particular solution, but there are many solutions, each with different values of a, b, and c. The general solution is a superposition of these,

$$\phi(x,y,z) = \sum_i \left(A_{1i}\mathrm{e}^{-a_ix} + A_{2i}\mathrm{e}^{a_ix}\right)\left(B_{1i}\mathrm{e}^{-b_iy} + B_{2i}\mathrm{e}^{b_iy}\right)\left(C_{1i}\mathrm{e}^{-c_iz} + C_{2i}\mathrm{e}^{c_iz}\right) \tag{2.3.6}$$

where $a_i^2 + b_i^2 + c_i^2 = 0$ for all i.

<u>Example</u>



Consider a U-shaped channel shaped as shown in the diagram. The channel extends infinitely along the $\hat{\mathbf{z}}$ axis. We have the following boundary conditions on the surfaces of the channel, $\phi = 0$ on the vertical sides, ϕ is a specified function on the bottom horizontal side, and $\phi \to 0$ as $y \to \infty$,

$$\phi(0,y) = 0, \qquad \phi(a,y) = 0, \qquad \phi(x,y) = 0 \text{ as } y \to \infty\,, \qquad \phi(x,0) = f(x) \text{ a specified function} \tag{2.3.7}$$

Because of the translational symmetry along the $\hat{\mathbf{z}}$ axis, the solution must be independent of z, so

$$\phi(x,y) = \sum_i \left(A_{1i}\mathrm{e}^{-a_ix} + A_{2i}\mathrm{e}^{a_ix}\right)\left(B_{1i}\mathrm{e}^{-b_iy} + B_{2i}\mathrm{e}^{b_iy}\right) \qquad \text{with} \qquad a_i^2 + b_i^2 = 0 \tag{2.3.8}$$

We can see that the correct thing is to choose a to be imaginary so that the dependence on y will exponentially decay as $y \to \infty$. We then write,

$$a_i = i\alpha_i, \qquad b_i = \alpha_i \qquad \text{so that } a_i^2 + b_i^2 = 0 \text{ is automatically satsified} \tag{2.3.9}$$

Using these we can write,

$$\phi(x,y) = \sum_i \left(A_i \cos\alpha_i x + B_i \sin\alpha_i x\right)\left(C_i \mathrm{e}^{-\alpha_i y} + D_i \mathrm{e}^{\alpha_i y}\right) \tag{2.3.10}$$

where

$$A_i = (A_{1i} + A_{2i}), \quad B_i = i(A_{1i} - A_{21}), \quad C_i = B_{1i}, \quad D_i = B_{2i} \tag{2.3.11}$$

Now $\phi(x,y) \to 0$ as $y \to \infty$ for all $x \quad \Rightarrow \quad \boxed{D_i = 0}$.

So now,

$$\phi(x,y) = \sum_i \left(A_i' \cos\alpha_i x + B_i' \sin\alpha_i x\right) \mathrm{e}^{-\alpha_i y} \tag{2.3.12}$$

where $A_i' = A_i C_i$ and $B_i' = B_i C_i$.

Now we use the boundary condition on the left vertical side at $x = 0$,

$$\phi(0,y) = 0 \quad \Rightarrow \quad \sum_i A_i' \mathrm{e}^{-\alpha_i y} = 0 \quad \text{for all } y \qquad \Rightarrow \qquad \boxed{A_i' = 0}. \tag{2.3.13}$$

So now,

$$\phi(x,y) = \sum_i B_i' \sin(\alpha_i x) \mathrm{e}^{-\alpha_i y} \tag{2.3.14}$$

Now we use the boundary condition on the right vertical side at $x = a$,

$$\phi(a,y) = 0 \quad \Rightarrow \quad \sum_i B_i' \sin(\alpha_i a) \mathrm{e}^{-\alpha_i y} = 0 \quad \text{for all } y \tag{2.3.15}$$

$$\Rightarrow \quad \sin(\alpha_i a) = 0 \quad \Rightarrow \quad \alpha_i a = n\pi \quad \Rightarrow \quad \alpha_i = \frac{n\pi}{a} \quad \text{for integer } n \geq 1 \tag{2.3.16}$$

So now,

$$\boxed{\phi(x,y) = \sum_{n=1}^{\infty} B_n' \sin\left(\frac{n\pi x}{a}\right) \mathrm{e}^{-n\pi y/a}} \tag{2.3.17}$$

Finally we use the boundary condition on the bottom horizontal side at $y = 0$,

$$\phi(x,0) = f(x) \qquad \Rightarrow \qquad \sum_{n=1}^{\infty} B_n' \sin\left(\frac{n\pi x}{a}\right) = f(x) \qquad \text{this is just the Fourier Series for } f(x)! \tag{2.3.18}$$

So we can determine the remaining unknown coefficients B_n' by the Fourier coefficient formula,

$$B_n' = \frac{2}{a}\int_0^a dx\, f(x) \sin\left(\frac{n\pi x}{a}\right) \tag{2.3.19}$$

The above follows from the orthogonality condition,

$$\frac{2}{a}\int_0^a dx\, \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases} \tag{2.3.20}$$

Just multiply Eq. (2.3.18) by $\sin\left(\frac{m\pi x}{a}\right)$, integrate over x, apply the orthogonality condition, and one gets Eq. (2.3.19).

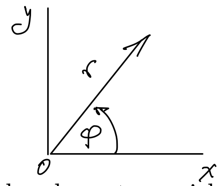
For the case of $f(x) = \phi_0$ a constant,

$$B_n' = \frac{2\phi_0}{a}\int_0^a dx\, \sin\left(\frac{n\pi x}{a}\right) = \frac{2\phi_0}{a}\left[\frac{-a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right]_0^a = \frac{2\phi_0}{n\pi}(1 - \cos n\pi) = \begin{cases} 0 & n \text{ even} \\ \dfrac{4\phi_0}{n\pi} & n \text{ odd} \end{cases} \tag{2.3.21}$$

Cylindrical Coordinates

As with the previous example, we will assume that our system contains no charge and has translational symmetry along the $\hat{\mathbf{z}}$ axis, so ϕ does not depend on the coordinate z. For $\phi(r,\varphi)$, where r is the cylindrical radial coordinate and φ the polar angle, Laplace's equation in cylindrical coordinates is,

$$\nabla^2\phi(r,\varphi) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} = 0 \tag{2.3.22}$$



We will assume a separation of variables solution, $\phi(r,\varphi) = R(r)\Phi(\varphi)$. Then we can write,

$$\frac{r^2\nabla^2\phi}{\phi} = \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = 0 \tag{2.3.23}$$

For the above to vanish at all r and φ, each term must be a constant,

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \nu^2 \qquad \text{and} \qquad \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -\nu^2 \tag{2.3.24}$$

so that the sum of the two terms is always zero. The solutions to the above are,

$$\begin{aligned} &R(r) = ar^\nu + br^{-\nu} \qquad &&\Phi(\varphi) = A\cos(\nu\phi) + B\sin(\nu\phi) \qquad &&\text{for } \nu \neq 0 \\ &R(r) = a_0 + b_0\ln r \qquad &&\Phi(\varphi) = A_0 + B_0\varphi \qquad &&\text{for } \nu = 0 \end{aligned} \tag{2.3.25}$$

How did we find these solutions? We just guess! Anytime a differential equation involves powerlaw terms and derivatives, an algebraic form or a polynomial is a good guess to try.

If our system is such that φ can take its entire range of values from 0 to 2π (such as a problem in which ϕ is specified on the surface of a cylinder) then ϕ must obey the periodicity $\phi(r,\varphi) = \phi(r,\varphi+2\pi)$. This requires that $B_0 = 0$ and $\nu = n$ an integer. In this case we have,

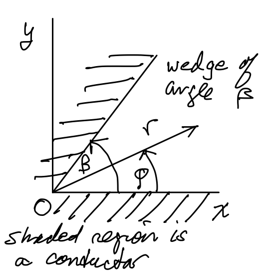
$$\phi(r,\varphi) = a_0 + b_0\ln r + \sum_{n=1}^{\infty}\left[r^n\left(A_n\cos n\varphi + B_n\sin n\varphi\right) + r^{-n}\left(C_n\cos n\varphi + D_n\sin n\varphi\right)\right] \tag{2.3.26}$$

or reparametrizing,

$$\phi(r,\varphi) = a_0 + b_0\ln r + \sum_{n=1}^{\infty}\left[a_n r^n\sin(n\varphi+\alpha_n) + b_n r^{-n}\sin(n\varphi+\beta_n)\right] \tag{2.3.27}$$

If the region where we are solving for ϕ includes the origin $r=0$ (suppose it is the region inside of the cylinder), then all the $b_n = 0$ since ϕ should not diverge at the origin if there is no charge there. If the region where we are solving for ϕ excludes $r=0$ (suppose it is the region outside the cylinder), then the b_n need not be zero. The case $b_0 \neq 0$ corresponds to a line charge λ along the $\hat{\mathbf{z}}$ axis.

Consider now the case where φ has a restricted range, for example a wedge shaped opening of angle β in a conducting block, so that φ is restricted to $0 \leq \varphi \leq \beta$.



ϕ is constant in the conductor, which gives the boundary conditions,

$$\phi(r,\varphi=0) = \phi_0 \qquad \phi(r,\varphi=\beta) = \phi_0 \tag{2.3.28}$$

The general solution is the linear combination

$$\phi(r,\varphi) = (a_0 + b_0\ln r)(A_0 + B_0\varphi) + \sum_{\nu>0}\left(a_\nu r^\nu + b_\nu r^{-\nu}\right)\left(A_\nu\cos\nu\varphi + B_\nu\sin\nu\varphi\right) \tag{2.3.29}$$

The boundary condition that $\phi(r,0) = \phi_0$ a constant for all r then requires,

$$\boxed{b_0 = 0, \qquad A_\nu = 0 \text{ for all } \nu} \tag{2.3.30}$$

So,

$$\phi(r,\varphi) = a_0(A_0 + B_0\varphi) + \sum_{\nu>0} \left(a_\nu r^\nu + b_\nu r^{-\nu}\right) B_\nu \sin\nu\varphi \tag{2.3.31}$$

Since ϕ should be continuous as one approaches the conducting surface, and $\phi = \phi_0$ is a finite constant on the conducting surface, then ϕ cannot diverge as one approaches the origin $r = 0$ along any fixed angle φ. This requires $\boxed{b_\nu = 0 \text{ for all } \nu}$.

So,

$$\phi(r,\varphi) = a_0(A_0 + B_0\varphi) + \sum_{\nu>0} a_\nu r^\nu B_\nu \sin\nu\varphi \tag{2.3.32}$$

The condition $\phi(r,\beta) = \phi_0$ a constant for all r then requires,

$$\sin\nu\beta = 0 \qquad \Rightarrow \qquad \nu = \frac{n\pi}{\beta}, \text{ with } n \text{ an integer } n \geq 1 \tag{2.3.33}$$

So,

$$\phi(r,\varphi) = a_0(A_0 + B_0\varphi) + \sum_{n=1}^{\infty} a_n r^{n\pi/\beta} \sin\left(\frac{n\pi\varphi}{\beta}\right) \tag{2.3.34}$$

Since ϕ must approach the constant ϕ_0 as $r \to 0$ along *any* fixed angle φ, we therefore must have

$$\boxed{B_0 = 0, \quad a_0 A_0 = \phi_0}. \tag{2.3.35}$$

So finally we have,

$$\phi(r,\varphi) = \phi_0 + \sum_{n=1}^{\infty} a_n r^{n\pi/\beta} \sin\left(\frac{n\pi\varphi}{\beta}\right) \tag{2.3.36}$$

But we still have all the unknown coefficients a_n! These will depend on how $\phi(r,\varphi)$ behaves as $r \to \infty$. We can't make the choice that $\phi \to 0$ as $r \to \infty$, because $\phi = \phi_0$ everywhere on the conducting surface even as $r \to \infty$. Thus we must have additional information if we are to determine the a_n.

Nevertheless, we can still get very interesting information near the origin at small r. In this limit, the leading term in the above series expansion for ϕ comes from the $n = 1$ term, as it vanishes the most slowly as $r \to 0$. So near the origin we can write,

$$\phi(r,\varphi) \approx \phi_0 + a_1 r^{\pi/\beta} \sin\left(\frac{\pi\varphi}{\beta}\right) \tag{2.3.37}$$

The radial and polar components of the electric field are then,

$$E_r(r,\varphi) = -\frac{\partial\phi}{\partial r} = -\frac{\pi a_1}{\beta}\, r^{\frac{\pi}{\beta}-1} \sin\left(\frac{\pi\varphi}{\beta}\right) \tag{2.3.38}$$

$$E_\varphi(r,\varphi) = -\frac{1}{r}\frac{\partial\phi}{\partial\varphi} = -\frac{\pi a_1}{\beta}\, r^{\frac{\pi}{\beta}-1} \cos\left(\frac{\pi\varphi}{\beta}\right) \tag{2.3.39}$$

Note, at $\varphi = 0$ or $\varphi = \beta$, we have $E_r = 0$ as it must, since the electric field must always be normal to the conducting surface.

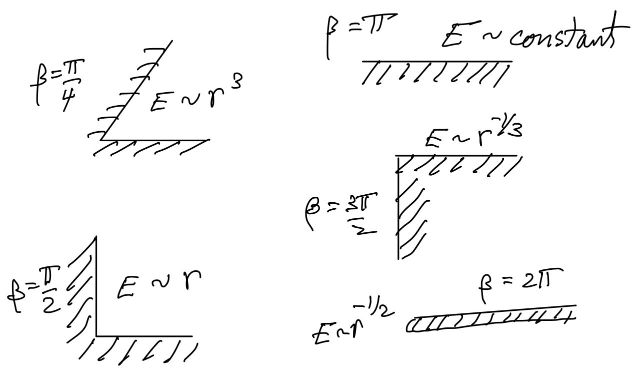
We thus onclude that as $r \to 0$, $\boxed{E \sim r^{\frac{\pi}{\beta}-1}}$.

The induced surface charge on the surface of the conductor is given by, $\mathbf{E} \cdot \hat{\mathbf{n}} = 4\pi\sigma$. For the surface at $\varphi = 0$, $\hat{\mathbf{n}} = \hat{\boldsymbol{\varphi}}$. For the surface at $\varphi = \beta$, $\hat{\mathbf{n}} = -\hat{\boldsymbol{\varphi}}$. We thus have,

$$\sigma(r, \varphi = 0) = \frac{E_\varphi(r, 0)}{4\pi} = -\frac{a_1}{4\beta} r^{\frac{\pi}{\beta}-1} \qquad \text{and} \qquad \sigma(r, \varphi = \beta) = \frac{-E_\varphi(r, 0)}{4\pi} = -\frac{a_1}{4\beta} r^{\frac{\pi}{\beta}-1} \tag{2.3.40}$$

For $\pi/\beta > 1$, i.e. $\beta < \pi$, $\mathbf{E}$ and σ vanish as $r \to 0$ and one approaches the origin.

For $\pi/\beta < 1$¡ i.e.$\beta > \pi$, $\mathbf{E}$ and σ diverge as $r \to 0$ and one approaches the origin.



Thus we conclude that $\mathbf{E}$ diverges at an *external* corner, while $\mathbf{E}$ vanishes at an *internal* corner. Remember, the above example had translational symmetry along the $\hat{\mathbf{z}}$ axis, so the "corners" are really infinitely long straight edges.

Our result here is an illustration of the general conclusion that electric fields diverge at sharp conducting corners. The same is true for a geometry where the conductor is a conical tip (though it is more involved mathematically to show it). This effect is the basis for the technologies of scanning tunneling microscopy and scanning force microscopy, where one scans a sharp metallic or semiconductor tip across a surface. In the first case, the strong electric field at the point of the tip serves to draw electrons off the surface; in the second case the strong electric field at the tip creates a force between the surface an the tip that deflects a cantilever. In both cases one uses these measurements to infer the properties and topology of the surface.

Spherical Coordinates

Finally we consider spherical coordinates, again for a region that contains no charge and so $\nabla^2\phi = 0$. You have probably already seen separation of variables in spherical coordinates when you solved the Schrodinger equation for the hydrogen atom in quantum mechanics. In spherical coordinates we can write Laplace's equation as,

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0 \tag{2.3.41}$$

and we then assume a solution of the form,

$$\phi(r, \theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi) \tag{2.3.42}$$

Then

$$\nabla^2\phi = 0 \quad \Rightarrow \quad r^2\nabla^2\phi = 0 \quad \Rightarrow \quad \Theta\Phi\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R\Phi}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{R\Theta}{\sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = 0 \tag{2.3.43}$$

$$\Rightarrow \quad \frac{r^2\sin^2\theta}{\phi}\nabla^2\phi = \frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = 0 \tag{2.3.44}$$

Note that the first two terms of the middle expression in the above depend only on the coordinates r and θ, while the last term depends only on the coordinate φ. For their sum to be zero for all values of r, θ and φ, it must be the case that the last term is a constant, while the sum of the first two terms is the negative of that constant. We'll call that constant $-m^2$. So we have,

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -m^2 \quad \Rightarrow \quad \frac{d^2\Phi}{d\varphi^2} = -m^2\Phi \quad \Rightarrow \quad \Phi(\varphi) = \mathrm{e}^{\pm im\varphi} \tag{2.3.45}$$

Since Φ must have 2π periodicity in φ, i.e. $\Phi(\varphi) = \Phi(\varphi + 2\pi)$, we have that $\boxed{m \text{ must be an integer}}$.

Returning to the other two pieces we have,

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = m^2 \tag{2.3.46}$$

divide all terms by $\sin^2\theta$ to get

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = 0 \tag{2.3.47}$$

The first term depends only on r, while the next two terms depend only on θ. For them to sum to zero, the first term must be a constant while the next two terms sum to the negative of that constant. We will call that constant $\ell(\ell+1)$.

We thus get for the radial part,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right) = \ell(\ell+1) \quad \Rightarrow \quad \frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right) = \ell(\ell+1)R \tag{2.3.48}$$

Since the differential equation involves powers of r and derivatives with respect to r, a good guess is to try a power law form for R. One finds that the solution has the form

$$\boxed{R(r) = a_\ell r^\ell + b_\ell r^{-(\ell+1)}} \tag{2.3.49}$$

We can substitute this into the differential equation to verify it is the solution,

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{d}{dr}\left(r^2\left[\ell a_\ell r^{\ell-1} - (\ell+1)b_\ell r^{-\ell-2}\right]\right) = \frac{d}{dr}\left(\ell a_\ell r^{\ell+1} - (\ell+1)b_\ell r^{-\ell}\right) \tag{2.3.50}$$

$$= \ell(\ell+1)a_\ell r^\ell + \ell(\ell+1)b_\ell r^{-(\ell+1)} = \ell(\ell+1)R \tag{2.3.51}$$

For the angular part we have,

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -\ell(\ell+1) \tag{2.3.52}$$

Let $x = \cos\theta$, so that $dx = -\sin\theta\,d\theta$, and $d\theta = -\dfrac{dx}{d\theta}$. Since θ takes the range $0 \le \theta \le \pi$, we have that x takes the range $-1 \le x \le 1$. The above differential equation then becomes,

$$\frac{d}{dx}\left[(1-x^2)\,\frac{d\Theta}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]\Theta = 0 \tag{2.3.53}$$

This is called the generalized Legendre Equation, and it has solutions on the range $-1 \le x \le 1$ when $\ell \ge 0$ is an integer. Those solutions are known as the *associated Legendre functions.*

For the special case $m = 0$, we have $\Phi(\varphi) = 1$ is independent of the azimuthal angle φ. Thus our solution $\phi(r,\theta,\varphi)$ does not depend on the angle φ, and the solution has rotational symmetry about the $\hat{\mathbf{z}}$ axis. For $m = 0$ we have,

$$\frac{d}{dx}\left[(1-x^2)\,\frac{d\Theta}{dx}\right] + \ell(\ell+1)\Theta = 0 \tag{2.3.54}$$

The solutions are known as the ordinary *Legendre polynomials*, $\mathcal{P}_\ell(x)$. For integer ℓ they are given by,

$$\mathcal{P}_\ell(x) = \frac{1}{2^\ell\,\ell!}\left(\frac{d}{dx}\right)^\ell \left(x^2-1\right)^\ell \qquad \text{Rodriguez's formula} \tag{2.3.55}$$

The polynomials for the lowest few ℓ are,

$$\mathcal{P}_0(x) = 1, \quad \mathcal{P}_1(x) = x, \quad \mathcal{P}_2(x) = \frac{1}{2}\left(3x^2-1\right), \quad \mathcal{P}_3(x) = \frac{1}{2}\left(5x^3-3x\right) \tag{2.3.56}$$

In general, $\mathcal{P}_\ell(x)$ is a polynomial of order ℓ with only even powers if ℓ is even, and only odd powers if ℓ is odd. Thus $\mathcal{P}_\ell(x)$ has the symmetry,

$$\mathcal{P}_\ell(x) = \mathcal{P}_\ell(-x) \quad \text{when } \ell \text{ is even,} \qquad\qquad \mathcal{P}_\ell(x) = -\mathcal{P}_\ell(-x) \quad \text{when } \ell \text{ is odd} \tag{2.3.57}$$

$\mathcal{P}_\ell(x)$ is normalized so that $\mathcal{P}_\ell(1) = 1$.

Note: The Legendre polynomials are the solutions only when ℓ is an integer and $\ell \geq 0$. One can wonder about solutions for non-integer ℓ. Also, for each integer ℓ, the Legendre polynomials give only one solution. But the differential equation that defines the $\mathcal{P}_\ell(x)$ is a 2nd order differential equation – 2nd order differential equations should have two solutions for each value of ℓ. So where are these "2nd" solutions, and where are the solutions for non-integer values of ℓ?

It turns out that all these other solutions blow up at either $x = -1$ or at $x = 1$, i.e. at $\theta = 0$ or $\theta = \pi$. They therefore are physically unacceptable for problems where θ is allowed to take its full range of values, $0 \leq \theta \leq \pi$, and where there is no reason why $\phi(r,\theta,\varphi)$ should be singular at either $\theta = 0$ or π. See Jackson section 3.2 for details.

The Legendre polynomials are *orthogonal* and form a *complete set of basis functions* on the interval $-1 \leq x \leq 1$,

$$\int_{-1}^{1} dx\,\mathcal{P}_\ell(x)\,\mathcal{P}_\ell(x) = \int_0^\pi d\theta\,\sin\theta\,\mathcal{P}_\ell(\cos\theta)\,\mathcal{P}_\ell(\cos\theta) = \begin{cases} 0 & \ell \neq m \\ \dfrac{2}{2\ell+1} & \ell = m \end{cases} \tag{2.3.58}$$

We can therefore expand any function $f(\theta)$ on the interval $0 \leq \theta \leq \pi$ as a linear combination of the $\mathcal{P}_\ell(\cos\theta)$. This is the reason they are useful for solving problems of Laplace's equation with spherical boundary conditions.

For the more general case when $m \neq 0$, the solutions to Eq. (2.3.53) are the associated Legendre functions $\mathcal{P}_\ell^m(x)$. For $\mathcal{P}_\ell^m(x)$ to be finite in the interval $-1 \leq x \leq 1$, one finds that ℓ must be an integer $\ell > 0$, and the integer values of m must statisfy $|m| \leq \ell$, i.e. $m = -\ell, -\ell+1, \ldots, 0, \ldots, \ell-1, \ell$. For each such ℓ and m there is only one non-divergent solution. See Jackson section 3.5 for details.

It is typical to combine the solutions $\mathcal{P}_\ell^m(\cos\theta)$ to the θ part of Laplace's equation with the $\Phi_m(\varphi) = \mathrm{e}^{im\varphi}$ solutions to the φ part, to define the *spherical harmonics*

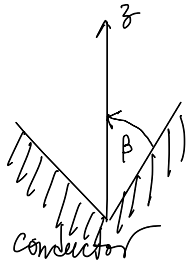
$$Y_{\ell m}(\theta,\phi) \equiv \sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\;\mathcal{P}_\ell^m(\cos\theta)\mathrm{e}^{im\varphi} \tag{2.3.59}$$

The $Y_{\ell m}(\theta,\phi)$ are orthogonal,

$$\int_0^{2\pi} d\varphi \int_0^\pi d\theta\,\sin\theta\;Y^*_{\ell' m'}(\theta,\phi)\,Y_{\ell m}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'} \tag{2.3.60}$$

where $Y^*_{\ell m}$ is the complex conjugate of $Y_{\ell m}$. They form a *complete set* of basis functions for expanding any function $f(\theta,\varphi)$ defined on the surface of a sphere.

<u>Behavior of fields near a conical hole or sharp tip</u>



Consider the geometry of a conical hole in a conductor, as in the diagram. The geometry has rotational symmetry about the $\hat{\mathbf{z}}$ axis. We want to solve $\nabla^2\phi = 0$ with separation of variables, but now θ is restricted to the range $0 \le \theta \le \beta$. We still have azimuthal symmetry, so this corresponds to the case $m = 0$ for the solution $\Phi_m(\varphi)$. But now, since we do not need the solution to be finite for all $0 \le \theta \le \pi$, but only for the range $0 \le \theta \le \beta$, we have to consider all the "other" solutions to the equation for $\Theta(\theta)$, i.e. ℓ no longer has to be integer, though one still needs $\ell \ge 0$ for the solution to be finite at $\theta = 0$. Similar to what we found for sharp edges in cylindrical coordinates, one finds that the resulting electric field $\mathbf{E} \to 0$ as $r \to 0$ when $\beta < \pi$, and $\mathbf{E} \to \infty$ as $r \to 0$ when $\beta > \pi$. See Jackson section 3.4 for details.

Examples with azimuthal symmetry: $m = 0$

When the problem has azimuthal symmetry, the general solution to $\nabla^2\phi = 0$ can be written as a linear combination of products of the $R_\ell(r)$ with the Legendre polynomials $\mathcal{P}_\ell(\cos\theta)$. This gives

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right]\mathcal{P}_\ell(\cos\theta) \tag{2.3.61}$$

The goal is to determine the coefficients A_ℓ and B_ℓ from the boundary conditions of the particular problem.

Example 1

Suppose one is given the value of the potential $\phi(R,\theta) = \phi_0(\theta)$ on the surface of a sphere of radius R.

Inside: To find the solution to $\nabla^2\phi = 0$ *inside* the sphere, we note that ϕ should not diverge at the origin, i.e. as $r \to 0$. This leads to the conclusion that $\boxed{B_\ell = 0 \quad \text{for all } \ell}$. Thus, inside,

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell \mathcal{P}_\ell(\cos\theta) \tag{2.3.62}$$

At the surface $r = R$ we must have

$$\phi_0(\theta) = \phi(R,\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell \mathcal{P}_\ell(\cos\theta) \tag{2.3.63}$$

To find the coefficients A_ℓ we multiply both sides by $\sin\theta\,\mathcal{P}_m(\cos\theta)$, integrate over θ, and apply the orthogonality condition of Eq. (2.3.58),

$$\int_0^\pi d\theta \sin\theta\,\phi_0(\theta)\,\mathcal{P}_m(\cos\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell \int_0^\pi d\theta \sin\theta\;\mathcal{P}_m(\cos\theta)\,\mathcal{P}_\ell(\cos\theta) \tag{2.3.64}$$

$$= \sum_{\ell=0}^{\infty} A_\ell R^\ell \left(\frac{2}{2\ell+1}\right)\delta_{\ell m} = A_m R^m \frac{2}{2m+1} \tag{2.3.65}$$

Thus

$$\boxed{A_m = \frac{2m+1}{2R^m}\int_0^\pi d\theta \sin\theta\;\phi_0(\theta)\,\mathcal{P}_m(\cos\theta)} \tag{2.3.66}$$

Outside: To find the solution to $\nabla^2\phi = 0$ *outside* the sphere, we can require that $\phi \to 0$ as $r \to \infty$. This leads to the conclusion that $\boxed{A_\ell = 0 \quad \text{for all } \ell}$. Thus, outside,

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}}\,\mathcal{P}_\ell(\cos\theta) \tag{2.3.67}$$

At the surface $r = R$ we must have

$$\phi_0(\theta) = \phi(R,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}}\,\mathcal{P}_\ell(\cos\theta) \tag{2.3.68}$$

We can solve for the coefficients B_ℓ with the same approach as we used above for the A_ℓ inside the sphere. We get,

$$\boxed{B_m = \frac{2m+1}{2}\,R^{m+1}\int_0^\pi d\theta\,\sin\theta\,\phi_0(\theta)\,\mathcal{P}_m(\cos\theta) \qquad \Rightarrow \qquad B_m = A_m R^{2m+1}} \tag{2.3.69}$$

<u>Example 2</u>

Suppose one is given the value of a fixed surface charge density $\sigma(\theta)$ on the surface of a sphere of radius R. What is the solution ϕ inside and outside the sphere?

From the previous example we know,

$$\phi(r,\theta) = \begin{cases} \displaystyle\sum_{\ell=0}^{\infty} A_\ell r^\ell\,\mathcal{P}_\ell(\cos\theta) & r < R \;\text{ i.e. inside} \\ \displaystyle\sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}}\,\mathcal{P}_\ell(\cos\theta) & r > R \;\text{ i.e. outside} \end{cases} \tag{2.3.70}$$

The boundary conditions at $r = R$ on the surface are now,

(i) ϕ is continuous

$$\Rightarrow \quad \phi^{\text{in}}(R,\theta) - \phi^{\text{out}}(R,\theta) = \sum_{\ell=0}^{\infty}\left[A_\ell R^\ell - \frac{B_\ell}{R^{\ell+1}}\right]\mathcal{P}_\ell(\cos\theta) = 0 \tag{2.3.71}$$

If an expansion in Legendre polynomials vanishes for all θ, then each coefficient in the expansion must vanish,

$$\Rightarrow \quad A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}} \quad \Rightarrow \quad \boxed{B_\ell = A_\ell R^{2\ell+1}} \tag{2.3.72}$$

(ii) The jump in the electric field is given by $\sigma(\theta)$

$$\left[-\frac{\partial\phi^{\text{out}}}{\partial r} + \frac{\partial\phi^{\text{in}}}{\partial r}\right]_{r=R} = 4\pi\sigma \tag{2.3.73}$$

$$\Rightarrow \quad \sum_{\ell=0}^{\infty}\left[\frac{(\ell+1)B_\ell}{R^{\ell+2}} + \ell A_\ell R^{\ell-1}\right]\mathcal{P}_\ell(\cos\theta) = \sum_{\ell=0}^{\infty}\left[\frac{(\ell+1)A_\ell R^{2\ell+1}}{R^{\ell+2}} + \ell A_\ell R^{\ell-1}\right]\mathcal{P}_\ell(\cos\theta) \tag{2.3.74}$$

$$= \sum_{\ell=0}^{\infty}(2\ell+1)R^{\ell-1}A_\ell\,\mathcal{P}_\ell(\cos\theta) = 4\pi\sigma(\theta) \tag{2.3.75}$$

Find the A_m by using the orthogonality condition of the Legendre polyonmials,

$$(2m+1)R^{m-1}A_m\left(\frac{2}{2m+1}\right) = 4\pi\int_0^\pi d\theta\,\sin\theta\,\sigma(\theta)\,\mathcal{P}_m(\cos\theta) \tag{2.3.76}$$

So,

$$\boxed{A_m = \frac{4\pi}{2R^{m-1}}\int_0^\pi d\theta\,\sin\theta\,\sigma(\theta)\,\mathcal{P}_m(\cos\theta)} \tag{2.3.77}$$

Example

Suppose $\sigma(\theta) = k\cos\theta$ on the surface of a sphere of radius R. What is ϕ inside and outside the sphere?

Note, since $\mathcal{P}_1(x) = x$, we can write $\sigma(\theta) = k\mathcal{P}_1(\cos\theta)$. Hence, by the orthogonality of the $\mathcal{P}_\ell(\cos\theta)$ we have that only $A_1 \neq 0$.

$$A_1 = \frac{4\pi}{2} k \int_0^\pi d\theta \sin\theta\, \mathcal{P}_1(\cos\theta)\mathcal{P}_1(\cos\theta) = \frac{4\pi}{2} k \left(\frac{2}{2+1}\right) = \frac{4}{3}\pi k \tag{2.3.78}$$

Using $B_1 = A_1 R^{2+1}$, we get,

$$\phi(r,\theta) = \begin{cases} \frac{4}{3}\pi k r\cos\theta & r < R \\ \frac{4}{3}\pi k \dfrac{R^3}{r^2}\cos\theta & r > R \end{cases} \tag{2.3.79}$$

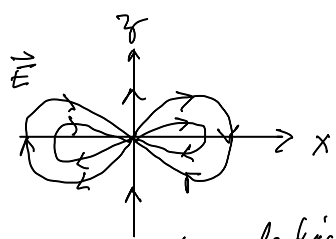
We will soon see that the potential outside the sphere is that of an ideal electric dipole with dipole moment $p = \frac{4}{3}\pi R^3 k$. Inside the sphere, the potential is $\phi = \frac{4}{3}\pi k z$, since $z = r\cos\theta$. The electric field inside the sphere is therefore constant,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi = -\frac{4}{3}\pi k\, \hat{\mathbf{z}} \tag{2.3.80}$$

Outside the sphere the field is,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi = -\frac{\partial\phi}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}} = \frac{8}{3}\pi k \frac{R^3}{r^3}\cos\theta\,\hat{\mathbf{r}} + \frac{4}{3}\pi k\frac{R^3}{r^3}\sin\theta\,\hat{\boldsymbol{\theta}} \tag{2.3.81}$$

$$= \frac{4}{3}\pi R^3 k \left[\frac{2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}}{r^3}\right] \qquad \text{we will soon see that this is just the field of an electric dipole} \tag{2.3.82}$$



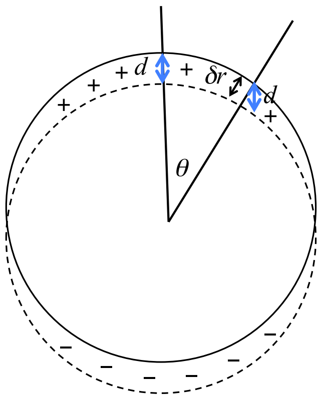
field lines of an electric dipole

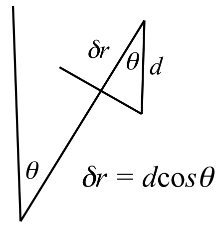
A physical example with $\sigma(\theta) = k\cos\theta$

Consider two spheres with equal radii R, with equal but opposite uniform charge densities ρ and $-\rho$. When the two spheres perfectly overlap, then the total $\rho_{\text{tot}} = 0$, and there is no net charge. No imagine displacing the positively charged sphere from the negatively charged sphere by a small distance d along the $\hat{\mathbf{z}}$ axis, $d \ll R$. This causes a surface charge $\sigma(\theta)$ to build up on the surface of the spheres, with a positive charge on the top, and a negative charge on the bottom.

If we think of each small element of charge in the positively charged sphere as being displaced by its corresponding element of charge in the negatively charged sphere, then this creates a small electric dipole moment. Considering this over the entire volume of the spheres, we see that this situation can be described as a *uniformly polarized* sphere, with a constant electric dipole density ρd throughout the sphere. We now want to compute the surface charge $\sigma(\theta)$ that results.

From the geometry as shown below, we see that the surface charge is $\sigma(\theta) = \rho\delta r = \rho d\cos\theta$.





Comparing to our previous example of a sphere with surface charge $\sigma = k\cos\theta$, we see that the uniformly polarized sphere is such an example with $k = \rho d$.

So

$$\boxed{\sigma(\theta) = \rho d\cos\theta}$$

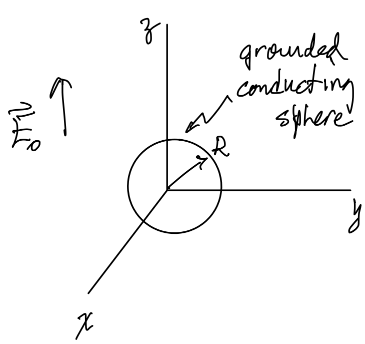
The total dipole moment on the sphere is $p = (\rho d)(\frac{4}{3}\pi R^3)$. The dipole moment density, also called the *polarization density* $\mathbf{P}$, is just the total dipole moment divided by the volume,

$$\mathbf{P} = p/(\frac{4}{3}\pi R^3) = \rho d$$

So, from our previous calculation, we conclude that the electric field $\mathbf{E}$ inside a sphere with uniform polarization $\mathbf{P}$ is constant, $\mathbf{E} = -\frac{4}{3}\pi k\,\hat{\mathbf{z}} = -\frac{4}{3}\pi\rho d\,\hat{\mathbf{z}} = -\frac{4}{3}\pi\mathbf{P}\,\hat{\mathbf{z}}$. We will come back to this result in our next unit.

<u>Example</u>

Consider a grounded conducting sphere of radius R placed in a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$. What is the resulting potential ϕ and what is the surface charge σ induced on the surface of the sphere?



As $r \to \infty$ far from the sphere, we should just have the uniform electric field, $\mathbf{E} = E_0\hat{\mathbf{z}} \quad\Rightarrow\quad \phi = -E_0 z = -E_0 r\cos\theta$. So the boundary conditions are:

$$\phi(R,\theta) = 0, \qquad \phi(r\to\infty,\theta) = -E_0 r\cos\theta \tag{2.3.83}$$

The solution outside the sphere has the form

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right]\mathcal{P}_\ell(\cos\theta) \tag{2.3.84}$$

Note, the A_ℓ are not necessarily zero here since ϕ does not vanish as $r\to\infty$. But applying the boundary condition as $r\to\infty$ we see that all $A_\ell = 0$ *except* $\boxed{A_1 = -E_0}$, since $\mathcal{P}_1(\cos\theta) = \cos\theta$. So

$$\phi(r,\theta) = -E_0 r\cos\theta + \sum_{\ell=0}^{\infty}\frac{B_\ell}{r^{\ell+1}}\mathcal{P}_\ell(\cos\theta) \tag{2.3.85}$$

Now apply the boundary condition on the surface, $\phi(R,\theta) = 0$ to get,

$$0 = -E_0 R\cos\theta + \sum_{\ell=0}^{\infty}\frac{B_\ell}{R^{\ell+1}}\mathcal{P}_\ell(\cos\theta) \tag{2.3.86}$$

From which we see that all $B_\ell = 0$ *except* for $\ell = 1$. For $\ell = 1$ we have $\dfrac{B_1}{R^2} = E_0 R \quad\Rightarrow\quad \boxed{B_1 = E_0 R^3}$.

(If two Legendre polynomial series are equal, then the coefficients of each term ℓ in the series must be equal. Since the first term in the above has only an $\ell = 1$ piece with coefficient $-E_0 R$, then only the $\ell = 1$ piece in the second term can be non-zero.)

So we get,

$$\boxed{\phi(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta} \tag{2.3.87}$$

The first term is just the potential $-E_0 r\cos\theta$ of the uniform applied electric field. The second term is the potential due to the induced surface charge on the surface of the sphere. We see it is a dipole field!

We can compute the induced charge density of the surface of the sphere,

$$4\pi\sigma(\theta) = -\left.\frac{\partial\phi}{\partial r}\right|_{r=R} = E_0\left(1+2\frac{R^3}{R^3}\right)\cos\theta = 3E_0\cos\theta \tag{2.3.88}$$

So

$$\sigma(\theta) = \frac{3}{4\pi}E_0\cos\theta \tag{2.3.89}$$

Looks just like a uniformly polarized sphere with $k = \dfrac{3E_0}{4\pi} = \mathbf{P}$. From the previous example we know that the field *inside* the sphere due to this $\sigma(\theta)$ is just

$$-\frac{4}{3}\pi k\,\hat{\mathbf{z}} = -\frac{4}{3}\pi\frac{3E_0}{4\pi}\,\hat{\mathbf{z}} = -E_0\,\hat{\mathbf{z}} \tag{2.3.90}$$

But that is just what we should have expected! The total field inside the conducting sphere must be zero. So the field created by the induced σ is just exactly the right amount to cancel out the applied field $E_0\hat{\mathbf{z}}$.

One can also check that just outside the surface of the sphere, the total electric field $\mathbf{E} = -\boldsymbol{\nabla}\phi$ is perpendicular to the surface, as it must be. I leave that to you as an exercise.