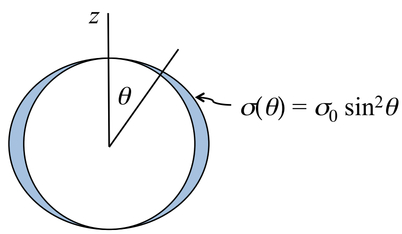
Unit 2-4-S: An Example



Suppose we have a sphere of radius R, and on the surface of the sphere is a fixed surface charge density

$$\sigma(\theta) = \sigma_0 \sin^2\theta \tag{2.4.S.1}$$

where θ is the usual spherical angle.

Let us find the solution for the electrostatic potential both inside and outside the sphere. Then we will compute the monopole, dipole, and quadrupole moments of this charge distribution and see that we can understand the exact solution outside the sphere in terms of the multipole expansion.

Exact Solution

To find the exact solution we will use separation of variable in spherical coordinates. Inside the sphere, $r < R$, since the potential should not blow up at the origin we know the solution must have the form,

$$\phi^{\rm in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \qquad r < R \tag{2.4.S.2}$$

Outside the sphere, $r > R$, since the potential should vanish as $r \to \infty$ we know the solution must have the form,

$$\phi^{\rm out}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \qquad r > R \tag{2.4.S.3}$$

The first boundary condition we use is that ϕ must be continuous as we cross a charged surface,

$$\phi^{\rm in}(R,\theta) = \phi^{\rm out}(R,\theta) \quad \Rightarrow \quad \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) \tag{2.4.S.4}$$

Just like with Fourier series, if two Legendre polynomial series are equal then all the Legendre coefficients must be equal, and so,

$$A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}} \qquad \Rightarrow \qquad \boxed{B_\ell = A_\ell R^{2\ell+1}} \tag{2.4.S.5}$$

The second boundary condition we use is that the normal component of the electric field $\mathbf{E} = -\boldsymbol{\nabla}\phi$ must jump by $4\pi\sigma$ as we cross a charged surface. In our case the normal vector is just the radial unit vector $\hat{\mathbf{r}}$, and so $\hat{\mathbf{n}} \cdot \mathbf{E} = -d\phi/dr$. We therefore have,

$$4\pi\sigma(\theta) = \left[-\frac{d\phi^{\rm out}}{dr} + \frac{d\phi^{\rm in}}{dr}\right]_{r=R} = \sum_{\ell=0}^{\infty}\left[(\ell+1)\frac{B_\ell}{r^{l+2}} + \ell A_\ell r^{\ell-1}\right]_{r=R} P_\ell(\cos\theta) \tag{2.4.S.6}$$

$$= \sum_{\ell=0}^{\infty}(2\ell+1)A_\ell R^{\ell-1} P_\ell(\cos\theta) \qquad \text{using Eq. (2.4.S.5)} \tag{2.4.S.7}$$

We can now solve for the A_ℓ using the orthogonality of Legendre polynomials. We could multiply both sides of the above by $P_m(\cos\theta)$ and integrate over $d\theta \sin\theta$, and use the orthogonality conditions. However, an easier way is to just write $\sigma(\theta)$ in terms of a linear combination of the P_ℓ, which we can do as follows. With $x \equiv \cos\theta$, and $\sin^2\theta = 1 - \cos^2\theta$ we have,

$$\sigma(x) = \sigma_0(1 - x^2) \tag{2.4.S.8}$$

Since this is a polynomial of order $n = 2$, we know that the expansion of $\sigma(x)$ in terms of the $P_\ell(x)$ can involve only the terms $\ell = 0, 1, 2$. With $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = \frac{1}{2}(3x^2 - 1)$ we have,

$$x^2 = \frac{2P_2 + 1}{3} = \frac{2P_2 + P_0}{3} \qquad \Rightarrow \qquad 1 - x^2 = P_0 - \frac{2P_2}{3} - \frac{P_0}{3} = \frac{2}{3}(P_0 - P_2) \tag{2.4.S.9}$$

so

$$\sigma(\theta) = \frac{2\sigma_0}{3}\left[P_0(\cos\theta) - P_2(\cos\theta)\right] \tag{2.4.S.10}$$

So now Eq. (2.4.S.7) becomes,

$$\frac{8\pi\sigma_0}{3}\left[P_0(\cos\theta) - P_2(\cos\theta)\right] = \sum_{\ell=0}^{\infty}(2\ell+1)A_\ell R^{\ell-1}P_\ell(\cos\theta) \tag{2.4.S.11}$$

and we see that all the $A_\ell = 0$ except for $\ell = 0$ and $\ell = 2$. For $\ell = 0$ the above gives,

$$\ell = 0: \qquad \frac{8\pi\sigma_0}{3} = A_0 R^{-1} \quad \Rightarrow \quad \boxed{A_0 = \frac{8\pi\sigma_0 R}{3} \quad \Rightarrow \quad B_0 = \frac{8\pi\sigma_0 R^2}{3}} \tag{2.4.S.12}$$

and for $\ell = 2$ we have,

$$\ell = 2: \qquad -\frac{8\pi\sigma_0}{3} = 5A_2 R \quad \Rightarrow \quad \boxed{A_2 = -\frac{8\pi\sigma_0}{15R} \quad \Rightarrow \quad B_2 = -\frac{8\pi\sigma_0 R^4}{15}} \tag{2.4.S.13}$$

So finally we have inside,

$$\boxed{\phi^{\text{in}}(r,\theta) = \frac{8\pi\sigma_0 R}{3} - \frac{8\pi\sigma_0}{15R}r^2 P_2(\cos\theta) = \frac{8\pi\sigma_0 R}{3} - \frac{8\pi\sigma_0 R}{30}\left(\frac{r}{R}\right)^2(3\cos^2\theta - 1) \qquad \text{for } r < R} \tag{2.4.S.14}$$

and outside we have,

$$\boxed{\phi^{\text{out}}(r,\theta) = \frac{8\pi\sigma_0 R^2}{3\,r} - \frac{8\pi\sigma_0 R^4}{15\,r^3}P_2(\cos\theta) = \frac{8\pi\sigma_0 R^2}{3\,r} - \frac{8\pi\sigma_0 R^4}{30\,r^3}(3\cos^2\theta - 1) \qquad \text{for } r > R} \tag{2.4.S.15}$$

Multipole Expansion

From the above exact solution for $\phi^{\text{out}}(r,\theta)$ we see that there are only $1/r$ and $1/r^3$ terms. These are the monopole and the quadrupole terms. Hence we know that the monopole and quadrupole moments are both non-zero, and the dipole moment and all moments higher than the quadrupole are zero.

But if we did not have the exact solution, we could still argue and calculate as follows.

Monopole

Since $\sigma(\theta) = \sigma_0 \sin^2\theta$, then we always have $\sigma(\theta) \geq 0$ (I assume $\sigma_0 > 0$), and hence when we integrate over the surface of the sphere we will get a positive answer. The net charge on the sphere must therefore be positive and so there is a non-zero monopole moment.

We can now compute the monopole moment, integrating over the surface of the sphere,

$$q = \int d^3r\,\rho(\mathbf{r}) = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta R^2 \sigma(\theta) = 2\pi\sigma_0 R^2 \int_0^{\pi} d\theta \sin^3\theta = \frac{8\pi\sigma_0 R^2}{3} \tag{2.4.S.16}$$

Where we used $\int_0^{2\pi} d\theta sin^3\theta = \int_0^{2\pi} d\theta(1-\cos^2\theta)\sin\theta = \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi} = (1-\frac{1}{3}) - (-1+\frac{1}{3}) = \frac{4}{3}$

The contribution to the potential from the monopole term is just q/r. Using the above q we see that we get exactly the first term in the expression of Eq. (2.4.S.15).

Dipole

Since the charge distribution $\sigma(\theta) = \sigma_0 \sin^2\theta$ has rotational symmetry about the $\hat{\mathbf{z}}$ axis (i.e. σ is independent of φ), if there was a dipole moment $\mathbf{p}$ it could not have any components in the xy plane, since all directions in the xy plane are equivalent by the rotational symmetry, so we must have $p_x = p_y = 0$. Suppose now that there was a finite p_z. By the behavior of vectors under a reflection we would get $p_z \leftrightarrow -p_z$ under reflection about the xy plane. But the charge distribution $\sigma(\theta)$ is reflection symmetric about the xy plane, i.e. $\sigma(\theta) = \sigma(\pi - \theta)$, which implies that p_z should not change under a reflection. The only consistent conclusion is that $p_z = 0$. We can thus conclude on the basis of symmetry that $\mathbf{p} = 0$ and there is no dipole moment. This agrees with our observation that there is no $1/r^2$ term in the exact potential $\phi^{\rm out}$ of Eq. (2.4.S.15).

If the symmetry argument is confusing for you, we could also explicitly calculate $\mathbf{p}$ and see that it is zero. For a point on the surface of the sphere we can write, $\mathbf{r} = (x, y, z) = R(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. We then have for the dipole moment,

$$\mathbf{p} = \int d^3r\,\rho(\mathbf{r})\mathbf{r} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta R^2 \sigma(\theta)\mathbf{r} = \sigma_0 R^3 \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin^3\theta \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \quad (2.4.S.17)$$

When we do the φ integration, the p_x component involves $\int_0^{2\pi} d\varphi \cos\varphi = 0$. Similarly, the p_y component involves $\int_0^{2\pi} d\varphi \sin\varphi = 0$. Hence $p_x = p_y = 0$. We see that these integrals vanish because $\sigma(\theta)$ is independent of φ, i.e. the charge distribution is rotationally symmetric about the $\hat{\mathbf{z}}$ axis.

The p_z component is independent of φ, and so does not vanish as we integrate over φ. But if we look at the θ integration we have, $\int_0^{\pi} d\theta \sin^3\theta\cos\theta = \left[\frac{1}{4}\sin^4\theta\right]_0^{\pi} = 0 - 0 = 0$. Hence $p_z = 0$. In this case it is the symmetry of $\sigma(\theta)$ about $\theta = \pi$, i.e. the reflection symmetry about the xy plane, that causes the integral to vanish.

We thus conclude that the dipole moment $\mathbf{p} = 0$.

Quadrupole

By rotation symmetry about the $\hat{\mathbf{z}}$ axis, we can infer that $Q_{xy} = Q_{yx} = Q_{xz} = Q_{zx} = Q_{yz} = Q_{zy} = 0$. The quadrupole tensor is therefore diagonal.

By rotation symmetry about the $\hat{\mathbf{z}}$ axis, we can also infer that $Q_{xx} = Q_{yy}$. And since $\overleftrightarrow{\mathbf{Q}}$ is traceless, we then have $Q_{xx} = Q_{yy} = -\frac{1}{2}Q_{zz}$. So the only element we really need to compute is Q_{zz}, and we can then get the entire quadrupole tensor $\overleftrightarrow{\mathbf{Q}}$.

But, just to see how the calculation goes, we will explicitly compute all the elements of $\overleftrightarrow{\mathbf{Q}}$ and see that they do indeed have the properties given above.

$$\overleftrightarrow{\mathbf{Q}} = \int d^3r\,\rho(\mathbf{r})\left(3\mathbf{rr} - r^2\overleftrightarrow{\mathbf{I}}\right) = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta R^2\sigma(\theta) \begin{pmatrix} 3x^2 - R^2 & 3xy & 3xz \\ 3xy & 3y^2 - R^2 & 3yz \\ 3xz & 3yz & 3z^2 - R \end{pmatrix} \quad (2.4.S.18)$$

where on the surface of the sphere $r^2 = R^2$. Now use $x = R\sin\theta\cos\varphi$, $y = R\sin\theta\sin\varphi$, and $z = R\cos\theta$ in the above to get,

$$\overleftrightarrow{\mathbf{Q}} = \sigma_0 R^4 \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin^3\theta \begin{pmatrix} 3\sin^2\theta\cos^2\varphi - 1 & 3\sin^2\theta\cos\varphi\sin\varphi & 3\sin\theta\cos\theta\cos\varphi \\ 3\sin^2\theta\cos\varphi\sin\varphi & 3\sin^2\theta\sin^2\varphi - 1 & 3\sin\theta\cos\theta\sin\varphi \\ 3\sin\theta\cos\theta\cos\varphi & 3\sin\theta\cos\theta\sin\varphi & 3\cos^2\theta - 1 \end{pmatrix} \quad (2.4.S.19)$$

When we compute the component Q_{xz} the integral over φ is $\int_0^{2\pi} d\varphi\, \cos\varphi = 0$, so $Q_{xz} = Q_{zx} = 0$. When we compute the component Q_{yz} the integral over φ is $\int_0^{2\pi} d\varphi\, \sin\varphi = 0$, so $Q_{yz} = Q_{zy} = 0$. When we compute the component Q_{xy} the integral over φ is $\int_0^{2\pi} d\varphi\, \cos\varphi\sin\varphi = \frac{1}{2}\int_0^{2\pi} d\varphi\, \sin(2\varphi) = 0$, so $Q_{xy} = Q_{yx} = 0$. We have thus confirmed that the quadrupole tensor is diagonal.

When we compute the component Q_{xx} the integral over the φ dependent piece is $\int_0^{2\pi} d\varphi \cos^2\varphi = \pi$. When we compute the component Q_{yy} the integral over the φ dependent piece is $\int_0^{2\pi} d\varphi \sin^2\varphi = \pi$. When we compute the component Q_{zz} there is no φ dependent piece and $\int_0^{2\pi} d\varphi = 2\pi$. So after doing the φ integration we get,

$$\overleftrightarrow{\mathbf{Q}} = \sigma_0 R^4 \int_0^\pi d\theta \sin^3\theta \begin{pmatrix} 3\pi\sin^2\theta - 2\pi & 0 & 0 \\ 0 & 3\pi\sin^2\theta - 2\pi & 0 \\ 0 & 0 & 6\pi\cos^2\theta - 2\pi \end{pmatrix} \tag{2.4.S.20}$$

So we have confirmed that $Q_{xx} = Q_{yy}$. Moreover we can see that when we compute $Q_{xx} + Q_{yy} + Q_{zz}$ the integrand is proportional to $6\pi\sin^2\theta + 6\pi\cos^2\theta - 6\pi = 6\pi - 6\pi = 0$, so we have confirmed that $\overleftrightarrow{\mathbf{Q}}$ is indeed traceless.

To complete the calculation we will need $\int_0^\pi d\theta \sin^3\theta = \frac{4}{3}$, $\int_0^\pi d\theta \sin^5\theta = \frac{16}{15}$, and $\int_0^\pi d\theta \sin^3\theta\cos^2\theta = \int_0^\pi d\theta \sin^3\theta(1 - \sin^2\theta) = \int_0^\pi d\theta(\sin^3\theta - \sin^5\theta) = \frac{4}{3} - \frac{16}{15} = \frac{4}{15}$. We thus get,

$$\overleftrightarrow{\mathbf{Q}} = \pi\sigma_0 R^4 \begin{pmatrix} \frac{3\cdot 16}{15} - \frac{2\cdot 4}{3} & 0 & 0 \\ 0 & \frac{3\cdot 16}{15} - \frac{2\cdot 4}{3} & 0 \\ 0 & 0 & \frac{6\cdot 4}{15} - \frac{2\cdot 4}{3} \end{pmatrix} = \pi\sigma_0 R^4 \begin{pmatrix} \frac{8}{15} & 0 & 0 \\ 0 & \frac{8}{15} & 0 \\ 0 & 0 & -\frac{16}{15} \end{pmatrix} = \frac{8\pi\sigma_0 R^4}{15} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{2.4.S.21}$$

We can now see explicitly that $Q_{xx} = Q_{yy} = -\frac{1}{2}Q_{zz}$.

Finally, the contribution of the quadrupole moment to the electrostatic potential is $\dfrac{\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}\cdot\hat{\mathbf{r}}}{2\,r^3}$. This gives a contribution to ϕ^{out} that is

$$\frac{8\pi\sigma_0 R^4}{30\,r^3}\,(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} \tag{2.4.S.22}$$

$$= \frac{8\pi\sigma_0 R^4}{30\,r^3}\,(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\cdot \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ -2\cos\theta \end{pmatrix} \tag{2.4.S.23}$$

$$= \frac{8\pi\sigma_0 R^4}{30\,r^3}\,\left(\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi - 2\cos^2\theta\right) = \frac{8\pi\sigma_0 R^4}{30\,r^3}\,\left(\sin^2\theta - 2\cos^2\theta\right) \tag{2.4.S.24}$$

$$= \frac{8\pi\sigma_0 R^4}{30\,r^3}\,\left(1 - 3\cos^2\theta\right) \tag{2.4.S.25}$$

This is exactly the second term in the exact solution for ϕ^{out} of Eq. (2.4.S.15). Thus the second term in Eq. (2.4.S.15) is just the quadrupole contribution. The exact solution tells us that all higher moments vanish.

Note: $\int_0^\pi d\theta \sin^5\theta = \int_0^\pi d\theta \sin\theta(1-\cos^2\theta)^2 = \int_0^\pi d\theta \sin\theta(1 - 2\cos^2\theta + \cos^4\theta) = \left[-\cos\theta + \frac{2}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta\right]_0^\pi = \left[1 - \frac{2}{3} + \frac{1}{5}\right] - \left[-1 + \frac{2}{3} - \frac{1}{5}\right] = \frac{8}{15} + \frac{8}{15} = \frac{16}{15}$