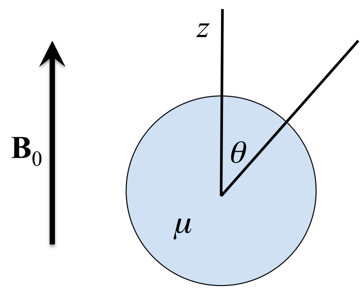
Unit 3-4-S: An Example



Suppose we have a sphere of magnetizable material of radius R and permeability μ. The sphere is in an external magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$. What are the magnetic fields inside and outside the sphere?

Since there is no free current in this problem the macroscopic $\mathbf{j} = 0$ and we can then use $\boldsymbol{\nabla} \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} = 0$ to write $\mathbf{H} = -\boldsymbol{\nabla}\phi_M$, where ϕ_M is the magnetic scalar potential for $\mathbf{H}$.

Note, since $\mathbf{B} = \mathbf{H}$ outside the sphere, and $\mathbf{B} = \mu\mathbf{H}$ inside the sphere, with μ spatially constant, we also have that $\boldsymbol{\nabla} \times \mathbf{B} = 0$ both inside and outside (but not necessarily at the boundary) and so we could have chosen a magnetic scalar potential $\tilde{\phi}_M$ for $\mathbf{B}$, so that $\mathbf{B} = -\boldsymbol{\nabla}\tilde{\phi}_M$. If you do things correctly, you will get the same results whether you work with ϕ_M or $\tilde{\phi}_M$; we will work with ϕ_M, the potential for $\mathbf{H}$.

Since inside the sphere we have a magnetic material, $\mathbf{B}^{\text{in}} = \mu\mathbf{H}^{\text{in}}$.

Since outside the sphere we have only a vacuum, $\mathbf{B}^{\text{out}} = \mathbf{H}^{\text{out}}$.

Now $\quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \boldsymbol{\nabla} \cdot \mathbf{H} = 0 \quad \Rightarrow \quad -\nabla^2\phi_M = 0$

and since the problem has rotational symmetry about the $\hat{\mathbf{z}}$ axis we know that we can write our solution for ϕ_M as a Legendre polynomial series. We can therefore write,

$$\underline{\text{Inside}}: \quad \phi_M^{\text{in}} = \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell(\cos\theta) \tag{3.4.S.1}$$

$$\underline{\text{Outside}}: \quad \phi_M^{\text{out}} = -B_0 z + \sum_{\ell=0}^{\infty} \frac{b_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \tag{3.4.S.2}$$

Note: in ϕ_M^{in} there are no $\dfrac{b_\ell}{r^{\ell+1}}$ terms since ϕ_M should not diverge as $r \to 0$. And in ϕ_M^{out} there are no $a_\ell r^\ell$ terms since ϕ_M^{out} must give only the external applied field at $r \to \infty$. The first term in ϕ_M^{out} just gives the external magnetic field $\mathbf{B}_0 = \mathbf{H}_0 = -\boldsymbol{\nabla}\phi_M = B_0\hat{\mathbf{z}}$, and we can rewrite this term as,

$$-B_0 z = -B_0 r\cos\theta = -B_0 r P_1(\cos\theta) \tag{3.4.S.3}$$

Boundary Conditions

i) The tangential component of $\mathbf{H}$ is continuous across an interface on which there is no macroscopic (i.e. free) sheet current, as is the case here. This is equivalent to saying that ϕ_M must be continuous at the surface of the sphere. This is completely analogous to the discussion in Notes 3-5 page 2, where we show that for $\mathbf{E} = -\boldsymbol{\nabla}\phi$, the fact that the tangential component of $\mathbf{E}$ is continuous at an interface implies that ϕ is continuous. So we have,

$$\phi_M^{\text{in}}(R,\theta) = \phi_M^{\text{out}}(R,\theta) \quad \Rightarrow \quad \sum_{\ell=0}^{\infty} a_\ell R^\ell P_\ell(\cos\theta) = -B_0 R P_1(\cos\theta) + \sum_{\ell=0}^{\infty} \frac{b_\ell}{R^{\ell+1}} P_\ell(\cos\theta) \tag{3.4.S.4}$$

Therefore, for $\ell \neq 1$ we have,

$$a_\ell R^\ell = \frac{b_\ell}{R^{\ell+1}} \quad \Rightarrow \quad \boxed{b_\ell = a_\ell R^{2\ell+1} \qquad \ell \neq 1} \tag{3.4.S.5}$$

while for $\ell = 1$ we have,

$$a_1 R = -B_0 R + \frac{b_1}{R^2} \quad \Rightarrow \quad \boxed{b_1 = (a_1 + B_0)R^3 \qquad \ell = 1} \tag{3.4.S.6}$$

ii) The normal component of $\mathbf{B}$ is continuous.

$$\mathbf{B}^{\text{in}} \cdot \hat{\mathbf{r}} = \mathbf{B}^{\text{out}} \cdot \hat{\mathbf{r}} \quad \Rightarrow \quad \mu \mathbf{H}^{\text{in}} \cdot \hat{\mathbf{r}} = \mathbf{H}^{\text{out}} \cdot \hat{\mathbf{r}} \quad \Rightarrow \quad \mu \frac{\partial \phi_M^{\text{in}}}{\partial r}\bigg|_{r=R} = \frac{\partial \phi_M^{\text{out}}}{\partial r}\bigg|_{r=R} \tag{3.4.S.7}$$

$$\Rightarrow \quad \mu \sum_{\ell=0}^{\infty} \ell\, a_\ell R^{\ell-1} P_\ell(\cos\theta) = -B_0 P_1(\cos\theta) - \sum_{\ell=0}^{\infty} \frac{(\ell+1) b_\ell}{R^{\ell+2}} P_\ell(\cos\theta) \tag{3.4.S.8}$$

Therefore for $\ell \neq 1$ we have,

$$\mu\, \ell a_\ell R^{\ell-1} = -\frac{(\ell+1) b_\ell}{R^{\ell+2}} \quad \Rightarrow \quad \boxed{b_\ell = -a_\ell\, \mu \frac{\ell}{\ell+1} R^{2\ell+1} \qquad \ell \neq 1} \tag{3.4.S.9}$$

while for $\ell = 1$ we have,

$$\mu\, a_1 = -B_0 - \frac{2b_1}{R^3} \quad \Rightarrow \quad \boxed{b_1 = -\frac{(\mu a_1 + B_0) R^3}{2} \qquad \ell = 1} \tag{3.4.S.10}$$

For $\ell \neq 1$ we need both Eqs. (3.4.S.5) and (3.4.S.9) to hold. This would imply that $-\mu \frac{\ell}{\ell+1} = 1$, which in general cannot be true; $\ell/(\ell+1) < 1$ and μ is usually positive. Hence we must conclude,

$$\boxed{a_\ell = b_\ell = 0 \qquad \ell \neq 1} \tag{3.4.S.11}$$

For $\ell = 1$ we need both Eqs. (3.4.S.6) and (3.4.S.10) to hold. This gives,

$$(a_1 + B_0) R^3 = -\frac{(\mu a_1 + B_0) R^3}{2} \quad \Rightarrow \quad \boxed{a_1 = -\frac{3B_0}{2+\mu}} \tag{3.4.S.12}$$

and substituting into Eq. (3.4.S.6) gives

$$\boxed{b_1 = \left(\frac{\mu - 1}{\mu + 2}\right) B_0 R^3} \tag{3.4.S.13}$$

We have now determined all the unknown constants, and so our solution is,

<u>Inside the sphere</u>:

$$\phi_M^{\text{in}} = a_1 r P_1(\cos\theta) = -\frac{3B_0}{2+\mu} r\cos\theta = -\frac{3B_0}{2+\mu} z, \qquad r < R \text{ inside} \tag{3.4.S.14}$$

and the magnetic field $\mathbf{H}$ inside is,

$$\mathbf{H}^{\text{in}} = -\boldsymbol{\nabla} \phi_M^{\text{in}} = \frac{3B_0}{2+\mu} \hat{\mathbf{z}} \qquad \text{is } \textit{uniform} \text{ inside the sphere} \tag{3.4.S.15}$$

the magnetic field $\mathbf{B}$ inside is,

$$\mathbf{B}^{\text{in}} = \mu \mathbf{H}^{\text{in}} = \frac{3\mu B_0}{2+\mu} \hat{\mathbf{z}} \tag{3.4.S.16}$$

and the magnetization density $\mathbf{M}$ inside the sphere is,

$$\mathbf{M} = \frac{\mathbf{B} - \mathbf{H}}{4\pi} = \frac{\mu - 1}{4\pi} \mathbf{H}^{\text{in}} = \frac{3(\mu-1)}{4\pi(\mu+2)} B_0 \hat{\mathbf{z}} \tag{3.4.S.17}$$

Assuming $\mu > 1$, the sphere thus has a paramagnetic response, with $\mathbf{M} \parallel \mathbf{B}_0$, and a uniform magnetization $\mathbf{M}$ with a total magnetic dipole moment,

$$\mathbf{m} = \frac{4\pi R^3}{3} \mathbf{M} = \left(\frac{\mu - 1}{\mu + 2}\right) B_0 R^3 \hat{\mathbf{z}} \tag{3.4.S.18}$$

Outside the sphere:

$$\phi_M^{\text{out}} = -B_0 z + \frac{b_1}{r^2}\, P_1(\cos\theta) = -B_0 z + \left(\frac{\mu - 1}{\mu + 2}\right) B_0 \frac{R^3}{r^2} \cos\theta, \qquad r > R \text{ outside} \tag{3.4.S.19}$$

and the magnetic field $\mathbf{B} = \mathbf{H}$ outside is,

$$\mathbf{B}^{\text{out}} = \mathbf{H}^{\text{out}} = -\boldsymbol{\nabla}\phi_M^{\text{out}} = -\frac{\partial \phi_M^{\text{out}}}{\partial r}\,\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial \phi_M^{\text{out}}}{\partial \theta}\,\hat{\boldsymbol{\theta}} = B_0\,\hat{\mathbf{z}} + \left(\frac{\mu - 1}{\mu + 2}\right) B_0 R^3 \left[\frac{2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}}{r^3}\right] \tag{3.4.S.20}$$

$$= B_0\,\hat{\mathbf{z}} + \mathbf{m}\left[\frac{2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}}{r^3}\right] \tag{3.4.S.21}$$

The magnetic field outside the sphere is just that of the applied magnetic field $\mathbf{B}_0$ plus the field of a pure magnetic dipole $\mathbf{m}$. Recall, we saw such a solution earlier when we discussed a uniformly magnetized sphere in Notes 3-3.