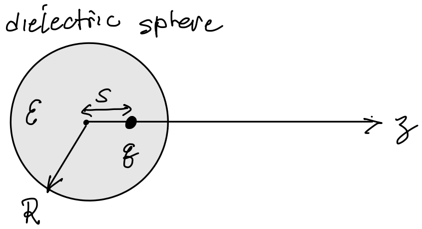
Unit 3-5: A More Interesting Example!

In the previous section we considered a point charge q at the center of a dielectric sphere of radius R. Now we want to consider a point charge q inside a dielectric sphere, but the charge is a distance s off from the center. What is the electrostatic $\mathbf{E}$ field inside and outside?



Inside we have $\boldsymbol{\nabla}\cdot\mathbf{D} = 4\pi\rho = 4\pi q\delta(\mathbf{r}-s\hat{\mathbf{z}})$.

$$\mathbf{D} = \epsilon\mathbf{E} \quad \Rightarrow \quad \boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho/\epsilon \tag{3.5.1}$$

For statics, we have,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi \quad \Rightarrow \quad \nabla^2\phi = -\frac{4\pi\rho}{\epsilon} = -\frac{4\pi q}{\epsilon}\delta(\mathbf{r}-s\hat{\mathbf{z}}) \tag{3.5.2}$$

The solution for $\phi(\mathbf{r})$ will be of the form,

$$\phi^{\text{in}}(\mathbf{r}) = \frac{q}{\epsilon|\mathbf{r}-s\hat{\mathbf{z}}|} + F(\mathbf{r}) \tag{3.5.3}$$

where the first term is due to the point charge q/ϵ (recall, the total charge at the location of q is the screened charge q/ϵ). The second term must satisfy $\nabla^2 F = 0$ and will be chosen to get the correct behavior at the boundary of the dielectric.

Since this problem has azimuthal symmetry (rotational symmetry about the $\hat{\mathbf{z}}$ axis), we can write F in terms of a Legendre polynomial expansion. For $r < R$ inside the sphere we must have,

$$F(\mathbf{r}) = \sum_{\ell=0}^{\infty} a_\ell\, r^\ell P_\ell(\cos\theta) \tag{3.5.4}$$

There are no $b_\ell/r^{\ell+1}$ terms since F should not diverge at the origin.

So inside, $r < R$, we have

$$\phi^{\text{in}}(\mathbf{r}) = \frac{q}{\epsilon|\mathbf{r}-s\hat{\mathbf{z}}|} + \sum_{\ell=0}^{\infty} a_\ell\, r^\ell P_\ell(\cos\theta) \tag{3.5.5}$$

From our discussion of the electric multipole expansion, we know that we can write for $r > s$,

$$\frac{1}{|\mathbf{r}-s\hat{\mathbf{z}}|} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{s}{r}\right)^\ell P_\ell(\cos\theta) \tag{3.5.6}$$

So for $r > s$ (but not for $r < s$) we have,

$$\phi^{\text{in}}(\mathbf{r}) = \sum_{\ell=0}^{\infty}\left[\frac{q}{\epsilon r}\left(\frac{s}{r}\right)^\ell + a_\ell\, r^\ell\right] P_\ell(\cos\theta) \qquad \text{for } s < r < R \tag{3.5.7}$$

Outside the sphere there is no charge, so $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$, or $\nabla^2\phi^{\text{out}} = 0$. So we can write,

$$\phi^{\text{out}}(\mathbf{r}) = \sum_{\ell=0}^{\infty}\frac{b_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \qquad \text{for } r > R \tag{3.5.8}$$

Outside there are no $a_\ell r^\ell$ terms since we must have $\phi^{\text{out}} \to 0$ as $r \to \infty$.

To determine the unknown a_ℓ and b_ℓ we use the boundary conditions at the surface of the dielectric at $r = R$.

1) The tangential component of $\mathbf{E}$ must be continuous

For a static situation where $\mathbf{E} = -\boldsymbol{\nabla}\phi$, at the surface of a dielectric (or interface between two different dielectrics), the condition that the tangential component of the electric field $\mathbf{E}_t$ is continuous as one crosses the surface is equivalent to the condition that ϕ is continuous. To see this, consider two points $\mathbf{r}_1$ and $\mathbf{r}_2$ on the surface. One then has,

$$\phi(\mathbf{r}_2) - \phi(\mathbf{r}_1) = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \boldsymbol{d\ell}\cdot\mathbf{E} \tag{3.5.9}$$

If we take the curve of integration to lie on the surface, then $\boldsymbol{d\ell}\cdot\mathbf{E}$ involves only the components of $\mathbf{E}$ that lie tangential to the surface. Since the tangential components of $\mathbf{E}$ are continuous, it does matter if we take the curve to lie just below the surface or just above the surface, we get the same result. So we conclude that for any two points on the surface,

$$\phi^{\text{below}}(\mathbf{r}_2) - \phi^{\text{below}}(\mathbf{r}_1) = \phi^{\text{above}}(\mathbf{r}_2) - \phi^{\text{above}}(\mathbf{r}_1) \tag{3.5.10}$$

So if the potentials ϕ^{above} and ϕ^{below} are equal at any one point on the surface, say $\phi^{\text{above}},(\mathbf{r}_1) = \phi^{\text{below}}(\mathbf{r}_1)$, then by the above they must be equal at all points on the surface. We can always add a constant to $\phi^{\text{below}}(\mathbf{r})$ so that it is equal to ϕ^{above} at a particular point, hence we conclude that $\phi^{\text{above}}(\mathbf{r}) = \phi^{\text{below}}(\mathbf{r})$ for all points $\mathbf{r}$ on the surface. This means $\phi(\mathbf{r})$ is continuous as one crosses the surface!

So in our present example,

$$\phi^{\text{in}}(R,\theta) = \phi^{\text{out}}(R,\theta) \qquad \Rightarrow \qquad \frac{q}{\epsilon R}\left(\frac{s}{R}\right)^\ell + a_\ell R^\ell = \frac{b_\ell}{R^{\ell+1}} \tag{3.5.11}$$

so,

$$\boxed{b_\ell = \frac{q}{\epsilon}s^\ell + a_\ell R^{2\ell+1}} \tag{3.5.12}$$

2) The normal component of $\mathbf{D}$ must be continuous (since the free surface charge is $\sigma = 0$)

Since $\hat{\mathbf{n}}\cdot\mathbf{D} = D_r$, and $\mathbf{D}^{\text{in}} = \epsilon\mathbf{E}^{\text{in}}$ and $\mathbf{D}^{\text{out}} = \mathbf{E}^{\text{out}}$, then

$$D_r^{\text{in}} = D_r^{\text{out}} \qquad \Rightarrow \qquad \epsilon E_r^{\text{in}} = E_r^{\text{out}} \qquad \Rightarrow \qquad -\epsilon\left.\frac{\partial\phi^{\text{in}}}{\partial r}\right|_{r=R} = -\left.\frac{\partial\phi^{\text{out}}}{\partial r}\right|_{r=R} \tag{3.5.13}$$

so,

$$\frac{(\ell+1)q}{R^2}\left(\frac{s}{R}\right)^\ell - \ell\epsilon a_\ell R^{\ell-1} = \frac{(\ell+1)b_\ell}{R^{\ell+2}} \qquad \Rightarrow \qquad qs^\ell - \frac{\ell}{\ell+1}\,\epsilon a_\ell R^{2\ell+1} = b_\ell \tag{3.5.14}$$

Now substitute in b_ℓ from Eq. (3.5.12) into the above to get,

$$qs^\ell - \frac{\ell}{\ell+1}\,\epsilon a_\ell R^{2\ell+1} = \frac{q}{\epsilon}s^\ell + a_\ell R^{2\ell+1} \tag{3.5.15}$$

Then solve for a_ℓ to get,

$$\boxed{a_\ell = \frac{qs^\ell}{R^{2\ell+1}}\,\frac{\left[1-\frac{1}{\epsilon}\right]}{\left[1+\left(\frac{\ell}{\ell+1}\right)\epsilon\right]}} \tag{3.5.16}$$

We can then plug this a_ℓ into Eq. (3.5.12) to get b_ℓ,

$$b_\ell = \frac{q}{\epsilon}s^\ell + a_\ell R^{2\ell+1} = \frac{q}{\epsilon}s^\ell + qs^\ell\frac{\left[1-\frac{1}{\epsilon}\right]}{\left[1+\left(\frac{\ell}{\ell+1}\right)\epsilon\right]} \tag{3.5.17}$$

After a bit of algebra, one gets,

$$\boxed{b_\ell = q s^\ell \left[\frac{1+\left(\frac{\ell}{\ell+1}\right)}{1+\left(\frac{\ell}{\ell+1}\right)\epsilon}\right]} \tag{3.5.18}$$

Whenever you get a solution, and the answer looks a bit complicated, it is always good if you can look at some simplified special cases where you know what the answer should be, and check that the complicated solution reduces to the known answer in that limit.

One such simplified special case is to take $s \to 0$, so that the charge lies at the center of the sphere, a problem we already solved in the previous section of the notes. In this limit, for $s = 0$,

$$a_\ell = b_\ell = 0 \quad \text{for all } \ell \neq 0 \tag{3.5.19}$$

$$a_0 = \frac{q}{R}\left[1 - \frac{1}{\epsilon}\right] \qquad \text{and} \qquad b_0 = q \tag{3.5.20}$$

So, for $s = 0$,

$$\phi^{\text{in}}(\mathbf{r}) = \frac{q}{\epsilon r} + \frac{q}{R}\left[1 - \frac{1}{\epsilon}\right] \qquad \Rightarrow \qquad \mathbf{E}^{\text{in}}(\mathbf{r}) = -\boldsymbol{\nabla}\phi^{\text{in}}(\mathbf{r}) = \frac{q}{\epsilon r^2}\,\hat{\mathbf{r}} \quad \text{as we found before.} \tag{3.5.21}$$

And,

$$\phi^{\text{out}}(\mathbf{r}) = \frac{q}{r} \qquad \Rightarrow \qquad \mathbf{E}^{\text{out}}(\mathbf{r}) = -\boldsymbol{\nabla}\phi^{\text{out}}(\mathbf{r}) = \frac{q}{r^2}\,\hat{\mathbf{r}} \quad \text{also as found before.} \tag{3.5.22}$$

Note, the constant, that is the second term in ϕ^{in} of Eq. (3.5.21), is just what is needed to make ϕ continuous at the surface $r = R$.

Another simplified case to check is to let $\epsilon \to \infty$. This should model a conductor! This is so since $\mathbf{D} = \epsilon\mathbf{E}$, so if $\epsilon \to \infty$, it must be that $\mathbf{E} \to 0$ so that $\mathbf{D}$ does not diverge, which is just what one has for a conductor, $\mathbf{E} = 0$ inside.

Again, for $\epsilon \to \infty$ one finds that $a_\ell = b_\ell = 0$ for all $\ell \neq 0$. For $\ell = 0$ one has,

$$a_0 = \frac{q}{R} \qquad \text{and} \qquad b_0 = q \tag{3.5.23}$$

so,

$$\phi^{\text{in}}(\mathbf{r}) = \frac{q}{\epsilon|\mathbf{r} - s\hat{\mathbf{z}}|} + \frac{q}{R} \quad \to \quad \frac{q}{R} \qquad \text{as } \epsilon \to \infty, \quad \text{and} \quad \mathbf{E}^{\text{in}} = -\boldsymbol{\nabla}\phi^{\text{in}} = 0 \tag{3.5.24}$$

So the field inside the sphere vanishes, and outside we have,

$$\phi^{\text{out}}(\mathbf{r}) = \frac{q}{r} \qquad \Rightarrow \qquad \mathbf{E}^{\text{out}} = -\boldsymbol{\nabla}\phi^{\text{out}} = \frac{q}{r^2}\,\hat{\mathbf{r}} \tag{3.5.25}$$

So the field outside is just like that of a point charge q at the origin, independent of where q is actually located inside the sphere. This is indeed just the behavior one would expect for a conducting sphere. The charge q on the conductor distributes itself as a uniform surface charge $\sigma = q/4\pi R^2$ on the surface, giving a zero field inside, and a field outside that looks like a point charge at the origin.