Unit 4: Electromagnetic Energy and Momentum

In this unit we extend the ideas of energy and momentum to electromagnetic fields, defining the energy density, energy current (Poynting vector), momentum density, and momentum current (Maxwell stress tensor) of the fields. We will derive conservation laws by which electromagnetic fields can exchange energy and momentum with mechanical degrees of freedom.

Unit 4-1: Electromagnetic Energy Density and the Poynting Vector

We will leave the macroscopic Maxwell equations for the present, and in this unit $\mathbf{E}$, $\mathbf{B}$, ρ, and $\mathbf{j}$ will refer to the exact microscopic quantities.

Consider a collection of charged particles, described by the charge density ρ and current density $\mathbf{j}$. The particles are contained within a volume V.

We define $\mathcal{E}_{\text{mech}}$ as the total "mechanical" energy of the particles. $\mathcal{E}_{\text{mech}}$ is the sum of the particles kinetic energy plus the potential energy of any non-electromagnetic forces.

The particles will exert forces on each other via their electromagnetic interactions, i.e., via the $\mathbf{E}$ and $\mathbf{B}$ fields that they create. Define W as the work done *on* the particles by all electromagnetic forces. Then, by the work-energy theorem of mechanics,

$$\frac{d\mathcal{E}_{\text{mech}}}{dt} = \frac{dW}{dt} \tag{4.1.1}$$

For a single charge q_i, and $\mathbf{F}_i$ the Lorentz force on the charge, we have,

$$\frac{dW}{dt} = \mathbf{F}_i \cdot \mathbf{v}_i = q_i \mathbf{E}(\mathbf{r}_i, t) \cdot \mathbf{v}_i + q_i \left(\frac{\mathbf{v}_i}{c} \times \mathbf{B}(\mathbf{r}_i, t) \right) \cdot \mathbf{v}_i = q_i \mathbf{E}(\mathbf{r}_i, t) \cdot \mathbf{v}_i \tag{4.1.2}$$

The term involving the magnetic field vanishes because $\mathbf{v}_i \times \mathbf{B}$ is orthogonal to $\mathbf{v}_i$.

For a collection of charges, with current density given by $\mathbf{j}(\mathbf{r}, t) = \sum_i q_i \mathbf{v}_i \delta(\mathbf{r} - \mathbf{r}_i(t))$, we have,

$$\frac{dW}{dt} = \sum_i q_i \mathbf{v}_i \cdot \mathbf{E}(\mathbf{r}_i, t) = \int_V d^3r \, \mathbf{j}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t) \tag{4.1.3}$$

as can be confirmed by substituting in the expression for $\mathbf{j}$ and integrating over the delta functions.

Now from Ampere's Law, $\quad \boldsymbol{\nabla} \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \quad$ we can write,

$$\mathbf{j} = \frac{c}{4\pi} \left[\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \right] \tag{4.1.4}$$

Substituting that into the above, we get

$$\frac{dW}{dt} = \int_V d^3r \, \mathbf{j} \cdot \mathbf{E} = \int_V d^3r \, \frac{c}{4\pi} \left[(\boldsymbol{\nabla} \times \mathbf{B}) \cdot \mathbf{E} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{E} \right] \tag{4.1.5}$$

To rewrite this expression, first note that,

$$\frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{E} = \frac{1}{2}\frac{\partial E^2}{\partial t}, \qquad \text{with} \qquad E^2 = |\mathbf{E}|^2 \tag{4.1.6}$$

Next,

$$\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) = (\boldsymbol{\nabla} \times \mathbf{E}) \cdot \mathbf{B} - \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) \quad \Rightarrow \quad \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) = (\boldsymbol{\nabla} \times \mathbf{E}) \cdot \mathbf{B} - \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) \tag{4.1.7}$$

then using

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Faraday's Law} \tag{4.1.8}$$

we have,

$$\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\cdot\mathbf{B} - \boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B}) = -\frac{1}{2c}\frac{\partial B^2}{\partial t} - \boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B}) \tag{4.1.9}$$

Substituting these into Eq. (4.1.5) we then get

$$\frac{d\mathcal{E}_{\rm mech}}{\partial t} = \frac{dW}{dt} = \int_V d^3r\,\mathbf{j}\cdot\mathbf{E} = -\frac{1}{4\pi}\int_V d^3r\left[\frac{1}{2}\frac{\partial E^2}{\partial t} + \frac{1}{2}\frac{\partial B^2}{\partial t} + c\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B})\right] \tag{4.1.10}$$

Now define,

$$\boxed{\begin{array}{ll} u \equiv \dfrac{1}{8\pi}\left(E^2+B^2\right) & \text{electromagnetic energy density} \\ \mathbf{S} \equiv \dfrac{c}{4\pi}\mathbf{E}\times\mathbf{B} & \text{Poynting vector = electromagnetic energy current} \end{array}} \tag{4.1.11}$$

Then we can write,

$$\frac{d\mathcal{E}_{\rm mech}}{dt} = -\int_V d^3r\left[\frac{\partial u}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S}\right] \quad\Rightarrow\quad \frac{d\mathcal{E}_{\rm mech}}{dt} + \frac{d}{dt}\int_V d^3r\,u = -\oint_{\mathcal{S}} da\,\hat{\mathbf{n}}\cdot\mathbf{S} \tag{4.1.12}$$

where $\mathcal{S}$ is the surface bounding V. Then, defining $\mathcal{E}_{\rm EM} \equiv \int_V d^3r\,u$ as the total electromagnetic energy in the volume V, and applying Gauss' Theorem to the integral of $\boldsymbol{\nabla}\cdot\mathbf{S}$, we have,

$$\boxed{\frac{d(\mathcal{E}_{\rm mech}+\mathcal{E}_{\rm EM})}{dt} = -\oint_{\mathcal{S}} da\,\hat{\mathbf{n}}\cdot\mathbf{S}} \tag{4.1.13}$$

The above is the reason for our identification of u as the electromagnetic energy density, and $\mathbf{S}$ as the electromagnetic energy current. The above says that the total energy in the volume V has two pieces: the mechanical energy of the charges $\mathcal{E}_{\rm mech}$, and a new piece $\mathcal{E}_{\rm EM}$ that represents the energy of the electromagnetic fields. If this is changing in time, and assuming that none of the charges are leaving the volume V, it can only be because electromagnetic energy is leaving the volume V. The right hand side therefore gives the flux of electromagnetic energy flowing out through the bounding surface $\mathcal{S}$; the minus sign is because a positive energy flux through $\mathcal{S}$ means energy is leaving V. Eq. (4.1.13) thus is the law of conservation of energy, when we include the energy of the electromagnetic fields.

We can also write the energy conservation law in differential form. If we define the mechanical energy density $u_{\rm mech}$ as the local mechanical energy per unit volume, so that $\mathcal{E}_{\rm mech} = \int_V d^3r\,u_{\rm mec}$, then we can write,

$$\frac{\partial(u_{\rm mech}+u)}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} = 0 \tag{4.1.14}$$

Note the similarity in form to the law of local charge conservation, $\partial\rho/\partial t + \boldsymbol{\nabla}\cdot\mathbf{j} = 0$.

Under certain simplifying conditions, we can derive a similar conservation law for the Macroscopic Maxwell Equations.

If we take $\mathbf{j}$ as the current density of the *free* charged particles, and $\mathbf{E}$ and $\mathbf{B}$ as the *macroscopic* electric and magnetic fields, then repeating the above steps we get,

$$\int_V d^3r\,\mathbf{j}\cdot\mathbf{E} = \frac{c}{4\pi}\int_V d^3r\left[\boldsymbol{\nabla}\times\mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}\right]\cdot\mathbf{E} \tag{4.1.15}$$

Writing

$$\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{H}\cdot(\boldsymbol{\nabla}\times\mathbf{E}) - \mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{H}) = -\frac{1}{c}\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} - \mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{H}) \tag{4.1.16}$$

we then get

$$\int_V d^3r\,\mathbf{j}\cdot\mathbf{E} = -\frac{1}{4\pi}\int_V d^3r\,\left[c\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right] \tag{4.1.17}$$

If the medium is *linear*, and we have *quasistatic* conditions, so that

$$\mathbf{D}(t) = \epsilon\mathbf{E}(t) \qquad \text{and} \qquad \mathbf{H}(t) = \frac{1}{\mu}\mathbf{B}(t) \qquad \text{with } \epsilon \text{ and } \mu \text{ constants,} \tag{4.1.18}$$

then we can write,

$$\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} = \epsilon\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} = \frac{\epsilon}{2}\frac{\partial E^2}{\partial t} = \frac{1}{2}\frac{\partial(\mathbf{E}\cdot\mathbf{D})}{\partial t} \tag{4.1.19}$$

and similarly,

$$\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} = \frac{1}{\mu}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} = \frac{1}{2\mu}\frac{\partial B^2}{\partial t} = \frac{1}{2}\frac{\partial(\mathbf{B}\cdot\mathbf{H})}{\partial t} \tag{4.1.20}$$

to get

$$\frac{d\mathcal{E}_{\text{mech}}}{dt} = \int_V d^3r\,\mathbf{j}\cdot\mathbf{E} = -\frac{d}{dt}\int_V d^3r\,u - \oint_{\mathcal{S}} da\,\hat{\mathbf{n}}\cdot\mathbf{S} \tag{4.1.21}$$

where $\mathcal{E}_{\text{mech}}$ is the mechanical energy of the free charges, and,

$$\begin{aligned} u &= \frac{1}{8\pi}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}\right) \qquad &&\text{is the electromagnetic energy density} \\ \mathbf{S} &= \frac{c}{4\pi}\mathbf{E}\times\mathbf{H} &&\text{is the macroscopic Poynting vector} \end{aligned} \tag{4.1.22}$$

Thus $\frac{1}{8\pi}\mathbf{E}\cdot\mathbf{D}$ is the electrostatic energy density, and $\frac{1}{8\pi}\mathbf{B}\cdot\mathbf{H}$ is the magnetostatic energy density.

Note, we said that the above result for the Macroscopic Maxwell Equations hold only under *quasistatic* conditions, i.e., when the fields are varying sufficiently slowly in time that ϵ and μ can be regarded as constants. This limitation arises because, as we will soon see, the dielectric "constant" ϵ is really not a constant at all, but will vary with frequency ω. As a consequence, we will see that $\mathbf{D}(t)$ and $\mathbf{E}(t)$ are *not* in general locally related in time, i.e., $\mathbf{D}(t) \neq \epsilon\mathbf{E}(t)$, but rather $\mathbf{D}(t) = \int dt'\,\tilde{\epsilon}(t-t')\mathbf{E}(t')$. Only when $\mathbf{E}$ is varying slowly in time, so that $\mathbf{E}$ only has Fourier components at small frequencies ω where $\epsilon(\omega) \approx$ constant, will we have the simpler $\mathbf{D}(t) \approx \epsilon\mathbf{E}(t)$.

Electrostatic Energy

Returning to the microscopic fields and charges, let's review the electrostatic energy as you may have first seen it in an earlier course. From the above, we have for the total electrostatic energy,

$$\mathcal{E} = \frac{1}{8\pi}\int_V d^3r\,E^2 \tag{4.1.23}$$

Now use $\mathbf{E} = -\boldsymbol{\nabla}\phi$ in electrostatics to get,

$$\mathcal{E} = -\frac{1}{8\pi}\int_V d^3r\,(\boldsymbol{\nabla}\phi)\cdot\mathbf{E} \tag{4.1.24}$$

Use $\boldsymbol{\nabla}\cdot(\phi\mathbf{E}) = \phi\boldsymbol{\nabla}\cdot\mathbf{E} + (\boldsymbol{\nabla}\phi)\cdot\mathbf{E}$ to get

$$\mathcal{E} = -\frac{1}{8\pi}\int_V d^3r\,\left[\boldsymbol{\nabla}\cdot(\phi\mathbf{E}) - \phi\boldsymbol{\nabla}\cdot\mathbf{E}\right] \tag{4.1.25}$$

Now use $\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho$, and Gauss' Theorem, to get

$$\mathcal{E} = -\frac{1}{8\pi}\oint_{\mathcal{S}} da\,\hat{\mathbf{n}}\cdot\phi\mathbf{E} + \frac{1}{2}\int_V d^3r\,\rho\phi \tag{4.1.26}$$

If the volume V expands to fill all space, so that the surface $\mathcal{S} \to \infty$, then we expect for localized charges that $\phi \sim 1/r$ and $E \sim 1/r^2$, and since the surface area grows as $\sim r^2$, we have that the surface integral $\sim r^2/r^3 \to 0$ as the length of the volume $r \to \infty$. We thus have,

$$\mathcal{E} = \frac{1}{2}\int d^3r\, \rho\phi \tag{4.1.27}$$

One can then also use the Coulomb solution, $\phi(\mathbf{r}) = \int d^3r'\, \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$, to write

$$\mathcal{E} = \frac{1}{2}\int d^3r \int d^3r'\, \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{4.1.28}$$

Note, Eq. (4.1.28) suggests the interpretation that $\mathcal{E}$ is due to an action-at-a-distance interaction between the charges. Eq. (4.1.27) suggests the interpretation that $\mathcal{E}$ is due to the potential energy of the charges ρ in the electrostatic potential ϕ, which suggests that the energy is stored in space at the locations where the charges are (since there is no contribution from regions where $\rho = 0$). In contrast, Eq. (4.1.23) suggests that the energy is stored in the electric field, and is located in space wherever $\mathbf{E} \neq 0$. These are all quite different interpretations. For example, for a uniformly charged sphere, Eq. (4.1.27) suggests that the energy is stored in the sphere, while Eq. (4.1.23) suggests the energy is stored throughout all space since $\mathbf{E} \neq 0$ everywhere.

Within electrostatics, there is no way to resolve which of these interpretations is correct, because they are all mathematically equivalent. We cannot ask where the energy is stored in a static situation – we can only ask such a question in a dynamic situation, when we can see how energy is transferred in time from one location to another. The derivation in the main part of this section thus shows that the correct equation for the electric energy, that holds in dynamic as well as static situations, is Eq. (4.1.23). The expressions of Eqs. (4.1.27) and (4.1.28) only hold in statics, but not more generally. Thus we conclude that the energy is stored in the electric field.

Magnetostatic Energy

We can do a similar analysis for the energy in magnetostatics. For the microscopic fields and currents, the total magnetostatic energy is,

$$\mathcal{E} = \frac{1}{8\pi}\int_V d^3r\, B^2 \tag{4.1.29}$$

Now use $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ to write,

$$\mathcal{E} = \frac{1}{8\pi}\int_V d^3r\, \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) \tag{4.1.30}$$

Use $\boldsymbol{\nabla}\cdot(\mathbf{B}\times\mathbf{A}) = \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) - \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A})$ to get,

$$\mathcal{E} = \frac{1}{8\pi}\int_V d^3r\, \left[\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) - \boldsymbol{\nabla}\cdot(\mathbf{B}\times\mathbf{A})\right] \tag{4.1.31}$$

And then use the *static* Ampere's Law, $\boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j}$, and Gauss' Theorem, to get,

$$\mathcal{E} = \frac{1}{2c}\int_V d^3r\, \mathbf{j}\cdot\mathbf{A} - \frac{1}{8\pi}\oint_{\mathcal{S}} da\, \hat{\mathbf{n}}\cdot(\mathbf{B}\times\mathbf{A}) \tag{4.1.32}$$

As we take V to fill all space, $\mathcal{S} \to \infty$, and for localized currents the surface integral will vanish. We are then left with,

$$\mathcal{E} = \frac{1}{2c}\int d^3r\, \mathbf{j}\cdot\mathbf{A} \tag{4.1.33}$$

In the Coulomb gauge $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$, we have $-\nabla^2\mathbf{A} = \frac{4\pi}{c}\mathbf{j}$ and the solution $\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int d^3r'\, \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$, and so substituting into the above we get,

$$\mathcal{E} = \frac{1}{2c^2}\int d^3r \int d^3r'\, \frac{\mathbf{j}(\mathbf{r})\cdot\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{4.1.34}$$

Eq. (4.1.34) gives the interpretation that the magnetostic energy is an action-at-a-distance interaction between currents. Eq. (4.1.33) gives the interpretation that the energy is stored where the current is, because there is no contribution from regions where $\mathbf{j} = 0$. Eq. (4.1.29) gives the interpretation that the energy is stored where the magnetic field is. As with the electrostatic energy, there is no way within statics to resolve which of these interpretations is correct. But by considering the more general dynamic situation, we see that it is Eq. (4.1.29) that remains correct in general, and hence the correct interpretation is that the magnetic energy is stored in the magnetic field.

Discussion Question 4.1

To derive Eq. (4.1.34) from Eq. (4.1.33) we explicitly used the Coulomb gauge for $\mathbf{A}$. Suppose we were *not* in the Coulomb gauge, i.e. $\boldsymbol{\nabla} \cdot \mathbf{A} \neq 0$. Would Eq. (4.1.34) continue to hold or not? Rather than just arguing in words, show mathematically that the value of the integral in Eq. (4.1.33) is *independent* of the choice of the gauge for $\mathbf{A}$, and hence Eq. (4.1.34) does not depend on our use of the Coulomb gauge.